MAR 1 1929

# THE MATHEMATICS TEACHER

THE OFFICIAL JOURNAL OF
THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

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Published by the
NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS
LANCASTER, PA. NEW YORK

Entered as second-class matter, March 26, 1927, at the Post Office at Lancaster, Pa., under the Act of March 3, 1879. Acceptance for mailing at special rate of postage provided for in Section 1103, Act of October 3, 1917, authorized November 17, 1921.

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THE MATHEMATICS TEACHER
525 West 120th St., New York City

SUBSCRIPTION PRICE $2.00 PER YEAR (eight numbers)
Foreign postage, 50 cents per year; Canadian postage, 20 cents per year. If remittance is made by check, five cents should be added for exchange.
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THE MATHEMATICS TEACHER

VOLUME XXII MARCH, 1929 NUMBER 3

# THE CULTURAL VALUE OF MATHEMATICS[1]

BY AUBREY J. KEMPNER

*University of Colorado, Boulder, Colorado*

## INTRODUCTION

A famous English mathematician and philosopher has defined mathematics as a science in which "we never know what we are talking about, nor whether what we say is true or false."—In all seriousness, there is some truth in this definition, but one may also maintain that it contains some exaggeration.

If mathematics, which we are accustomed to consider the most pure and most logically developed science, offers such difficulties when we attempt to define it, what shall we expect of such a relatively vague term as "culture"? Indeed, I doubt whether a satisfactory definition of culture can be given. In order to gain a common ground for our discussion, I shall not attempt to give a formal definition at all, but shall, instead, assume that everything is of cultural value which makes life fuller and richer; which places us in a position of greater harmony with our surroundings; which leads us to a better understanding of nature, of our individual development and of the development of the race.

In speaking of the cultural value of mathematics, I emphatically disclaim any intention of establishing a thesis that mathematics is of greater cultural value than any one of half a dozen other sciences. I am convinced that true culture would imply a universally high level of understanding of all sciences and arts. Unfortunately, the time has passed when it was possible for one human mind to acquire encyclopedic knowledge of science as a

[1] Read at the meeting of the Colorado Education Association, November 10, 1928, Pueblo and Denver, Colorado.

whole, or even to acquire complete mastery of any single science. The eighteenth century may have witnessed the last serious efforts in this direction. The average college graduate of the present day must be satisfied with an extremely thin veneer in all branches of knowledge except a very few. On these few his best efforts must be concentrated.

I have no desire to argue, for example, the relative merit of a training in history, in the ancient languages, in arts, or in mathematics. A person who is devoid of all understanding of the history of our civilization, who is ignorant of the classical spirit, who has no feeling for the influence which the fine arts have had on our development, is surely every bit as uncultured as one possessing no knowledge of mathematics. I shall only attempt to show, in a very inadequate manner, that mathematics does have a message of fundamental importance, that it does make a tremendous contribution to the possibility of orienting oneself in nature and in life.

### The General Cultural Influence of Mathematics

I do not belong to those who argue that science is science only insofar as it is dominated by pure logic. For certain, no science must contain anything which is contrary to the laws of logic; but most sciences have manifold roots and methods, and deductive logic is in many cases only one of them. Each science generally consists of two integral parts, which may perhaps be described as follows:

#### I. *Phenomenological* (*Descriptive*)

For example, in History, Botany, Zoology this part is very prominent. From a vast amount of observed data certain laws are distilled by incomplete induction (evolution, laws of heredity, and the like), but they are usually not really proven by deductive logic; instead, our minds are crushed into submission by the weight of empirical evidence. Yet, who will deny that we are, in such branches of knowledge, dealing with true science?

#### II. *Logical* (*Deductive*)

On the other hand, in Physics, in Chemistry, in Engineering, etc., we often come across laws which may, under certain fun-

damental assumptions, be considered logically necessary consequences of observed data. Whenever such laws are expressible in terms of space, numbers, and time, we are dealing with laws for which mathematics affords the natural means of treatment.

In the various sciences, these two methods of development are represented in varying proportions.

May I give for the second type a simple illustration of which I am particularly fond. Assume that we have a force emanating from an isolated point in space, and emanating uniformly in all directions from the point. Assume further that there is no absorption of this force. It follows then from quite elementary geometrical considerations that the intensity of the force is inversely proportional to the square of the distance. We need only to apply the theorem that in similar plane figures the areas are as the squares of corresponding lengths. Note that it was not even necessary to make any assumption concerning the mode of propagation, whether in straight lines or in concentric spherical shells. We have here the true reason why so many different types of forces (gravitational, electric, magnetic, light, heat, sound) all obey the same law of the inverse square of the distance.

Now consider the headlight of an automobile, in clear weather, where absorption may be neglected. It seems to blind us nearly as strongly when we stand in the path of the beam at a distance of a quarter of a mile as at a distance of one hundred yards. What is the explanation of this? If the former law held, the pupil of our eye would receive in the first case only about one twentieth of the amount it receives in the second case. If we assume a true parabolic reflector, it follows from geometry that the same amount of light is received by the pupil at a greater as at a smaller distance; hence the intensity is independent of the distance. Here you have laws of physics, some of them fundamental, not indeed proven, because it still remains to be shown that the assumptions hold; but how simple the assumptions appear when contrasted with the conclusions! To decide whether the assumptions actually are true or not lies outside of the domain of mathematics. The refined theory of relativity indicates that they are true to within a nearly incredible degree of accuracy, but that they are not quite exact. Do not these childishly simple examples show that mathematics gives us a rather penetrating insight into the mysteries of nature?

It is characteristic of mathematics as a logical science that it starts out with a small number of fundamental assumptions (axioms and postulates) which we are willing to accept without further proof, because they appeal to our common sense as obviously true (for example $a + b = b + a$, when we are dealing with numbers), and then, by applying to these assumptions the ordinary laws of logic, the whole science is developed; and, as far as it is developed, it will stand as long as our minds obey the apparently inexplorable laws of logic!

Let us ask ourselves whether this in itself does not contain elements of cultural value? I feel confident that I am not claiming too much for mathematics when I say that this concept of a system of deductive logic, our confidence in the infallibility of the application of our formal laws of logic to an empirically given situation, would never have taken hold of the mind of mankind to the extent that we trust in it as firmly as we trust in anything in the universe, if the human mind had not been trained in thousands of years of mathematical reasoning. Pray do not interpret my words wrongly: I do not claim that mathematics is logic, or that logic is mathematics. Both of these positions have been defended at various times, with unsatisfactory results. I do claim that mathematics is the outstanding test for our system of logic, the only science of which we may say, if I am allowed to plagiarize a trademark, that it is 99.44 percent pure. In mathematics, the remaining 0.56 percent are not impurities, but are occupied by the axioms and postulates.

This is no small matter! To acquire unreserved confidence in the autonomous power of our human mind means that we have gained the firm fundament on which all deductive science may be built.

Then, have you ever considered what it means to the maturing mind of a student that he is working in a branch of science where his own reasoned logical conclusions cannot be challenged by any outside authority, neither by instructor nor by textbook? I still remember with pleasure a student of a particular class in calculus a dozen years ago who asked me in an examination whether one problem could not be worked in a new manner. I told him that he would surely run into trouble. His characteristic reply was a delightful: "Well, I'm going to try, anyway." And he did, and he did! He spoiled his paper, but I

have always felt that he richly deserved the "100" I gave him. To me, one of the delights of teaching mathematics in a university is this knowledge that no student has to accept any statement or proof on my authority; instructor and student are on the same plane, and a bluff on the part of the instructor is apt to be called at any moment. In a country as proud of democracy as are these United States, this ought to have a certain power of appeal.

Of course, every science offers to a lesser or greater degree the opportunity for training independence of thought, and it goes without saying that in all teaching it is fundamental to cultivate this ability to the utmost. In the natural sciences it is generally the observed fact which the pupil must be trained to recognize for himself. May I cite in this connection one horrible example of how not to teach? In a university course in botany, my daughter could not see certain cells under the microscope appearing in the shape the instructor had indicated. When she very politely insisted that the cells looked different to her, she was silenced by the amazing retort: "Miss K., if I were you, I wouldn't argue with me!"

Next, I should like to emphasize the æsthetic joy which we experience in a structure of pure logic, flawlessly put together, aspiring to great heights, and yet firmly rooted in the soil of logic.

Thus, we have already encountered three most important elements:

I. *The growth of confidence in the power of the human mind.*
II. *The training of independence of thought.*
III. *Appreciation of a closely woven system of thought processes.*

It would be impossible to deny that these give the human mind a firm foundation, a starting point from which it may hope to comprehend the world in which it has its existence.

The fact that mathematics, together with perhaps chemistry and music, possesses the most powerful symbolism of any science or art, forms one of the important aspects of mathematics. One of the sources of its strength lies here. On the other hand, it means that what amounts to a new language must be learned for its mastery. To appreciate the importance of this symbolism, it is sufficient to imagine that you had to carry in words in your mind such a comparatively simple formula as

$$C = (d^2y/dx^2) \cdot \{1 + (dy/dx)^2\}^{-3/2}$$

or

$$(a + b)^n = a^n + \frac{n}{1}a^{n-1}b + \frac{n(n-1)}{1.2}a^{n-2} \cdot b^2 + \cdots + \frac{n}{1}ab^{n-1} + b^n.$$

Too many people think they are learning mathematics while they are merely memorizing the new language. As well to memorize words from a French dictionary and fondly imagine that you are learning the French language, or to learn the logarithm table by heart, to master trigonometry! (I may mention that there is, in a certain state university, a legend of a Chinese student who told his trigonometry instructor that he enjoyed the course very much, but that he did find it *very* hard to memorize the tables. He had only succeeded in learning six pages so far!) The formulæ represent only a façade of our mathematical mansion, or, if you like, a scaffolding. I might even go so far as to say that mathematics must be understood in spite of the formulæ. Unless you venture behind the formulæ, unless you realize that the formulæ themselves do not represent the thought processes of mathematics, but only the stations between thought processes and then the final result, so long "hast Du Seines Geistes keinen Hauch verspürt."

We owe to the power of the symbolism of mathematics, as well as to a natural sifting process which is in each generation weeding out the useless contributions of former generations, the fact that we are not crushed under the accumulated load of more than two thousand years of mathematical research. To read critically all mathematical papers which appear at the present time would more than absorb the total time of any individual, so that we should be in serious danger of finding ourselves in the position of the gifted student of one of the foremost mathematicians of the nineteenth century, *Jacobi:* It is reported that this pupil could not be persuaded to do any original work, because he felt obliged to first read all existing mathematics. Jacobi's argument against this attitude was: "Will you never marry because you do not have time to meet all the girls in the world, before choosing?"

These ideas help us to see more clearly the meaning of the term "science of mathematics." During each generation we are

tempted to consider valuable every contribution which is new, original. From a higher viewpoint, we consider valuable in mathematics only what contributes organically to the science as a whole. Anything that leads us into a blind alley, that does not point the path to further progress may be amusing, may be interesting, but will usually not endure through the centuries. The feeling of reverence which the mathematician often feels toward his science is greatly enhanced by this sense of compactness, of elimination of what is not organically necessary.

This development of the power to distinguish between what is fundamental in a science and what is really of minor importance, although it may at first appear more interesting than the basic factors, comes very late in the mathematical career of a student. In fact, it is one of the features hardest to acquire. However, it is so valuable that I shall assign to it a separate place in our schedule, and say:

IV. *Training to recognize fundamental features of a science, or of a situation.*

One more word about our symbolism! It is open to argument whether we should consider it a blessing or a curse that the different races of the earth separated before their spoken and written language had crystallized, so that we live among a myriad of tongues instead of possessing one universal language. But we surely have reason to rejoice that in mathematics and in some other sciences and arts we have well-nigh universal languages. To-day, the formula for $(a+b)^n$ conveys the same meaning, without further explanation, to the American, German, French, as to the Hindu, Japanese, Chinese mathematician. Is there not in this something of cultural importance?

Permit me a last remark in this direction. I am reasonably free of the narrow doctrine of Caucasian superiority over all colored races. I admire Chinese civilization, as far as I understand it. I regard the ancient Sanskrit writings (in translations, of course) with proper reverence. I realize that the Hindus had attained to a peak of civilization at a time when dark barbarism governed Nordic Europe. Yet, in spite of personal acquaintance with educated Hindus I held a conviction that present day Hinduism represented only a faint afterglow of a glory long departed. Alas for the sneaking sense of racial superiority which I still held! Like a meteor from the sky, a little

over a decade ago, a young pure blood Hindu mathematician, Ramanujan, astounded European mathematicians by developing new methods for attacking some of the most difficult theorems in that most abstract field of mathematics, the theory of numbers. Practically self-tutored in an Indian village, going through an Indian university with only the narrowest contact with Europeans, sullen, suspicious, afraid to divulge his new methods for fear they might be stolen, he suddenly stood forth as the peer of leading scientists of the world. We saw with illuminating clearness that the creative genius of the race which perhaps gave to the world its wonderful notation for writing numbers is not extinguished.

To us, to-day, it is obvious that we can write down numbers as large as we like, that is, our notation is powerful enough to express any finite integer, by simply taking a sufficient number of digits. It is hard for us to realize that Archimedes had difficulty in devising a notation which could represent any very large number. He did this in his famous "grains of sand" problem. If you do not quite sense the immense importance of a powerful system of notation, I invite you to multiply the two numbers MDCLXXVII and CCCLXIV without translating them into the Hindu-Arabic system.

In Japan, as well as in China and India, there are to-day mathematicians who are internationally known.

The roots of our present day mathematics go back to the ancient Egyptians, Assyrians, Chinese, Hindus, to the Arabs, to the marvelous Greeks; and we have lately unearthed evidence that South American Indians, before the discovery of America, apparently isolated from contact with any other civilization, had developed a very remarkable mathematics.

It must be of great interest even to people who have enjoyed a minimum of college training in mathematics to trace through more than twenty centuries such a topic as the problem of algebraic solution of equations. Starting with the Greek and Hindu and Arabic methods for dealing with linear and quadratic equations, written in a system of symbols which appears to us now-a-days absurdly cumbersome, passing on to the efforts of Cardan, Tartaglia et al., Italian mathematicians of the sixteenth century, who succeeded in solving the general third and fourth degree equations, then farther through more than

two centuries of heroic and unavailing efforts to solve the general fifth degree equation by algebraic methods, until in the early nineteenth century two of the most tragic figures in the history of mathematics, the young Norwegian Nils Henrik Abel and the young Frenchman Evariste Galois succeeded in showing that the general fifth degree and higher degree equations are not algebraically solvable.

The history of mathematics has become a large branch of science in its own right. In Europe and in this country live a number of well known mathematicians who devote their whole time and energy to it.

Does it not strike you that a realization of the development of a science, embracing a span of thirty centuries and races from all parts of the world may contain elements of considerable cultural value? Let me then add:

V. *Mathematics is a common heritage from many races and many ages.*

## Some Mathematical Concepts and Their Cultural Value

I have talked up to the present in very general terms. The remainder of my time I shall devote to more specific topics.

If we ask for a few general notions from mathematics which give us insight into problems arising in the extraneous world, and which are at the same time of sufficient generality to merit some degree of understanding even by people not interested in mathematics in a technical sense, I should like to mention the ideas of: *functionality, continuity, a derivative as a rate of change, and an integral as a summation process.* We will discuss these notions very briefly in the order indicated.

1. *Functionality.* We have a very simple function when we write $y = x^2$, or $\psi(x) = x^2$. We may assign to $x$ any value, then $y = x^2$ gives us a corresponding functional value. The graphical representation of this function as a parabola is very familiar. Always, an ordinary function may be represented by a graph. This is of great help in a vast number of applications of mathematics.

Why do we attach to this notion of functionality such great importance? Because it expresses our mathematical way of describing a law! It represents, as far as mathematics is concerned, a relation of causality, of cause and effect: *if* $x = 3$,

*then* $y=9$; *if* $x=a$, *then* $y=a^2$. The law itself may have come from a mathematical problem ($A=\pi r^2$), or, it may be derived from physics ($s=\frac{1}{2}gt^2$), or from chemistry ($p=c/v$), or from any field of science or engineering, etc., where we are able to establish quantitative relations governed by some law. You are familiar with the many attempts which have been made to let fundamental college courses in mathematics center around this notion of functionality. Whenever we deal with a law which is capable of mathematical representation, it translates itself into a function in the hands of the mathematician.

2. *Continuity.* In everyday language, when we say a process is continuous, we mean that it has no sudden break. If a stone drops, or a wheel turns, the motion is continuous; if a gas expands it must do so continuously. The feeling for this continuity in natural processes led already in the middle ages to the saying: "Natura non facit saltus." On the other hand, your price for a railroad ticket does not change continuously; it jumps suddenly by at least one cent, but probably by four cents. Nor does a taxi-fare change continuously. In nature, however, unless you delve into the subatomic world, probably all processes, when interpreted in a natural fashion, are continuous. It seems therefore quite inevitable that the notion of continuity must be another fundamental idea. Unfortunately, as happens so frequently, what represents the most natural way of expressing this notion in mathematical form does not appeal to the layman as particularly simple: If $y=f(x)$ is continuous at $x=x_0$, then $\lim_{x=0}\{f(x_0+\Delta x)-f(x_0)\}=0$, where it would require several hours to satisfactorily explain to a non-mathematician the meaning of the prefixed $\lim_{\Delta x=0}$.

3. *Derivative.* The derivative of the calculus usually represents for the uninitiated the very height of mathematical ambition. Farther than that he dare not look, and I suspect that many a student has regretted looking so far. Already the definition

$$\frac{df(x)}{dx}=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x},$$

whether written as a formula, or read off in words more or less well sounding, is undoubtedly impressive. (Incidentally I call

your attention to the close relation between the mathematical definitions of continuity and of a derivative.) Let us see whether the derivative is really as terrifying as it appears. The derivative $df(x)/dx$ of a function expresses simply the rate of change of the function $f(x)$ as compared with the rate of change of the variable $x$. For example, if $f(x) = 3x$, so that for $x = 1$, $f(x) = 3$, for $x = 2$, $f(x) = 6$, then $f(x)$ is changing three times as fast as $x$ is changing, and $d(3x)/dx = 3$. Now it may seem of little importance whether we know these rates of change of functions or not; but I assure you that nothing has been of greater importance in the whole history of mathematics, and, possibly, in the whole history of the sciences, than the introduction of this calculus by the Englishman Sir Isaac Newton and the German G. W. Leibnitz, in the latter half of the seventeenth century. An entire new world was opened to the mathematician and the physicist. Let me illustrate by using the simplest examples. Suppose that we have a train moving with a uniform speed of forty miles an hour. What does this mean? In $t$ hours the train has traveled $s = 40t$ miles. The velocity, 40 mi./hr., is nothing but the rate of change of the distance $s$ as compared with the change of time, $ds/dt = 40$ mi./hr. This simplest case, where we are dealing with uniform motion, can still be completely understood without the use of limiting processes, which are characteristic of the calculus. But now consider a body falling from rest. The distance $s$ through which it falls in $t$ seconds is given by $s = \frac{1}{2}gt^2$, as we know from physics. If we divide the total distance by the total time, we have of course $s/t = \frac{1}{2}gt$, the average velocity of the falling body. However, this average velocity is attained at just one instant of the whole process; before this instant the average velocity is greater than the actual velocity, afterwards smaller. How shall we obtain the true velocity at any instant? This is accomplished in the following natural manner. Instead of considering the whole time during which the body falls, imagine this time broken up into small time intervals $\Delta t$, for example into tenths of seconds. During any one of these tenths of seconds, the velocity will change but little although it may vary markedly in the different tenths. If the velocity be considered as not changing in one of these small time intervals, we have approximately uniform motion during this short period, and

during this period the velocity will be given by dividing the distance the body falls by the length of the time interval. The smaller the time interval is chosen, the smaller the error. By passing over to the limit, as one says, we obtain the true velocity at any given moment. Now the process just outlined is traced mathematically exactly, step for step, by taking the derivative of $s = \frac{1}{2}gt^2$,

$$\frac{ds}{dt} = v = \lim_{\Delta t = 0} \tfrac{1}{2}g \cdot \frac{(t + \Delta t)^2 - t^2}{\Delta t} = gt.$$

I crave your pardon for having dealt in some detail with so familiar an example. One excuse for doing so is that I have repeatedly found extremely hazy notions in this respect even in candidates for a doctor's degree in physics, with mathematics as a first minor. A second, better, reason is that such rates of change occur continually in nature. Their examination is one of the most important problems we have to deal with in our study of the physical laws of nature. Without explaining the statement in detail, I will say that the derivative very often gives us a tool for penetrating from the phenomenon to the underlying law, as in our case, where we pass from the observed phenomenon $s = \frac{1}{2}gt^2$ to the underlying law $v = gt$.

4. *Integration.* Integration is defined as the operation inverse to differentiation; that is, given the rate of change of a variable quantity, to determine the quantity itself. In applications, the fundamental importance of the integration process is apt to show up in an entirely different manner. Let us go back to our problem of the falling body, and assume this time that we know the speed to be given by the same law as before, $v = gt$. We now consider unknown the distance $s$ through which the body falls during the time $t$, and ask whether we can derive the value $s = \frac{1}{2}gt^2$ from the known expression for the speed. You see the problem is the converse of the one considered above. Again, let the whole time be divided into small intervals $\triangle t$, of, for example, one tenth of a second each. As before we may approximately consider the body falling with constant velocity through any one of these little intervals. For any one such interval the distance fallen through will therefore equal this constant velocity times the length of the interval, with great approximation. Adding all of these products together we shall

have a sum which is written

$$\sum_{t=0}^{t=T} gt \cdot \Delta t,$$

and this gives us a fair approximation to the total distance fallen through from the time $t=0$ to $t=T$. This approximate expression for the distance $s$ goes over into the true value of $s$ when we perform a limiting process, as we say, letting each little time interval $\Delta t$ approach 0. We are thus led to the limit of a summation process,

$$\lim_{\Delta t=0} \sum_{t=0}^{t=T} gt \cdot \Delta t = \int_{t=0}^{t=T} gt \cdot dt = \tfrac{1}{2}gt^2 = s.$$

(I call particular attention to the close parallelism in the symbolism employed to represent the sum and the integral.) We thus see that, starting from the velocity $v=gt$ as given, we obtain the distance $s=\frac{1}{2}gt^2$ in the form of an integral, while when we start from the distance $s$, the velocity $v$ is obtained as a derivative. This will at least make plausible the fact that integration really represents an operation inverse to differentiation. Again, without further amplifying my statement, I will merely say that the integration process permits us, in many cases, to derive from the underlying law ($v=gt$) the observable phenomenon ($s=\frac{1}{2}gt^2$).

In both operations, in differentiation as well as in integration, we have to deal with a so-called limiting process ($\Delta t$ approaches 0, in our examples). These limiting processes form, as already stated, the distinctive feature of calculus as contrasted with elementary mathematics—not with entire justification, as one recognizes when one considers the elementary "proofs" of the formulæ $C=2\pi r$, and $v=4\pi r^3/3$ and $s=4\pi r^2$ for the circumference of a circle and for the volume and surface of a sphere, respectively. It is, however, generally conceded that "higher mathematics" starts with the calculus. Whenever we are dealing with a formula in mathematics, or with motion, or with any process in nature which is capable of representation by a formula, in other words, whenever we are dealing with a function in mathematics, or with a law in physics, we are, except in the very simplest cases (uniform motion and linear functions), driven to calculus for any deeper investigation.

In looking back, are we claiming too much when we say that the four notions which I have so inadequately outlined represent, on account of their pervading generality, on account of their power of accommodation which enables them to deal with the most heterogeneous groups of phenomena from widely divergent fields, most powerful instruments for understanding the world we live in, and for penetrating to the very heart of the secrets of nature?

Whether we deal with our vast universe, of which the solar system is but a very humble member, and our earth a totally insignificant speck, or whether we deal with the systems of molecules and atoms, of inconceivably small dimensions and yet each representing a complete little universe, or whether finally we deal with terrestrial objects of dimensions neither too gigantic nor too minute to be grasped by our senses, they are all dominated by the same laws of mathematics, and they all seem to yield to the same general types of treatment.

Mathematics constitutes the most powerful means we possess to reduce the apparent chaos of phenomena in nature to at least a semblance of order; it creates law from anarchy, and while, to be sure, we have no guarantee that the type of order which we, by the aid of mathematics, project into nature really describes the innermost causal connections between the phenomena, it does tend to create for us an harmonious picture of the universe, a picture which has at least a definite relation to the deeper absolute reality which from the nature of our mind, must ever remain hidden from us.

## Beauty in Mathematics

In these concluding remarks, I do not wish to dwell on the relations between mathematics and the arts, the keepers of the beautiful, important though these relations be. It is, indeed, not too much to say that some of our arts, such as drawing, and, if you are willing to count it among the arts, architecture, are firmly based on mathematical principles. Of the latter, this is obvious; for the former, I need only remind you of the laws of perspective. In music, too, we have strong mathematical elements. One thinks immediately of Bach's well tempered scale, dividing the octave into twelve tones, with a constant ratio of vibrations from one tone to the next, and leading to a geometri-

cal progression of ratio $\sqrt[12]{2}$. Instead, I wish to emphasize the beauty of mathematics itself.

I am sure we all agree that the appreciation of the beautiful, whether in art, in nature, or in science, contains elements of vast cultural value. It is my impression that this aspect has not been sufficiently emphasized in connection with mathematics. It is taken for granted, of course, that mathematics will appear beautiful to him who devotes his life to this exacting mistress, but it also seems to be assumed that the beauties of this science are by their nature hidden from the eyes of the layman. In a certain sense, this is undoubtedly true. There can be no greater thrill than is experienced by the searcher when he discovers a new truth, created by him, a child of his own mind. The creative artist, as well as the productive scientist, knows full well that these rare moments represent the supreme heights of life. To quote from Kipling's *The Light that Failed:* The young painter, Dick, who knew that he was going blind, did not think of the doom that was to overtake him, but *"was swallowed up in the clean, clear joy of creation, which does not come to man too often, lest he should consider himself the equal of his God, and so refuse to die at the appointed time."* These exquisite thrills are not entirely denied to the enthusiastic amateur in mathematics, since there are numberless isolated theorems in elementary mathematics waiting to be born, most of them not of supreme importance, to be sure, and very few only destined to survive the centuries, yet of great beauty, and which would not require any technical training beyond the first year of college mathematics. May I be permitted to mention one of my own earliest experiences? I shall never forget the intense joy that filled my soul when I discovered, as a high school student, the following absurdly simple geometrical theorem concerning any two non-intersecting circles or, any two circles, the center of each of which does not lie in the other circle: $A_1A_2 = B_1B_2$. Here was something which I had created my very self, something which I knew, which I possessed, which, as I fondly thought, no other human mind shared with me, a secret truth my own in a truer sense than any material object could be my own. What matter that the theorem is nothing but a tiny flower blooming at the feet of its gorgeous sisters which in their totality constitute the glorious garden of the mathematical sci-

ences! Across the years, I still look back on that little flower with tender feelings.

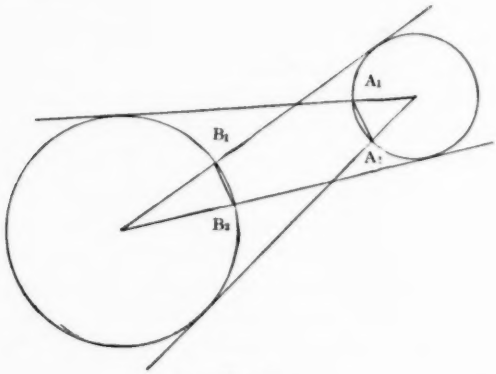


FIG. 1

The professional mathematician has come to regard with considerable respect contributions to the elementary parts of the science. A theorem may be interesting and important although it deals with simple questions, and many of the leading mathematicians of the world have contributed toward the development of elementary mathematics. But if we eliminate the delight that accompanies original discoveries, is there not something left for those who learn without creating? I am tempted to go so far as to say that there is a wealth of beauty in mathematics which can be appreciated by those who are not capable of proving the theorems or of even following the formal proof when it is presented to them. We respond to a Beethoven Sonata even though we be not able to read notes; we love a beautiful flower even though botany be to us a sealed mystery; the majestic swing of the Iliad thrills us even though we know nothing of the technical rules of rhythm. I should like to maintain that a vast number of mathematical theorems have this same degree of fascination, through their sheer æsthetic beauty, their simplicity of statement and richness of content. Even if we do not feel inclined to go to the extreme of Pythagoras who twenty-five centuries ago is said to have sacrificed a hecatomb of oxen to properly express his exultation at having discovered the theorem in geometry that goes under his name, and although the theorem may have been spoiled for many of you by poor peda-

gogical methods, do you not feel the beauty of a poem in the theorem: In every right triangle the square over the hypotenuse is exactly equal to the sum of the squares over the other two sides, $c^2 = a^2 + b^2$? For those whose pleasure is marred by unhappy recollections, I put down, without further explanation, two simple figures illustrating the lovely so-called Hindu proof of the theorem.

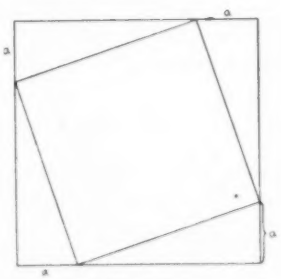

FIG. 2a

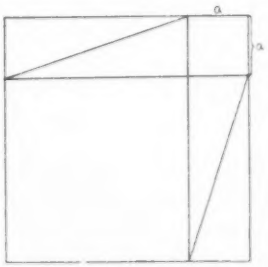

FIG. 2b

If properly taught, I believe every high school student will experience a sensation of encountering something beautiful when he learns that in every triangle the altitudes, or the medians, or the bisectors of the angles, or the perpendicular bisectors of the sides, meet in one point.

Not always is a beautiful theorem, even though the statement may be very simple, easily proven. For example, the thrilling theorem that every positive integer is either itself a square, or the sum of two squares, or the sum of three squares, or the sum of four squares defied all attempts at proof for over a century. The branch of mathematics which is known as the theory of numbers abounds in problems which are very easily stated and have been verified in thousands of cases, but which have never been proven in spite of heroic efforts of the best minds, stretching over many centuries.

Another charming theorem, this time from the very purest branch of geometry, projective geometry, and which is remarkable on account of the extreme degree of freedom which one is allowed in the choice of the lines used, is the following: Given any two straight not parallel lines, $l_1$ and $l_2$, and a point $A$ not on either of them. Draw any four lines through $A$ and complete the figure as indicated. Then the lines $l_1$, $l_2$, $l_3$ always intersect

in the same point. This may be used to construct a line which, prolonged, would pass through the point of intersection of two given lines, when the point of intersection falls outside the drawing board.

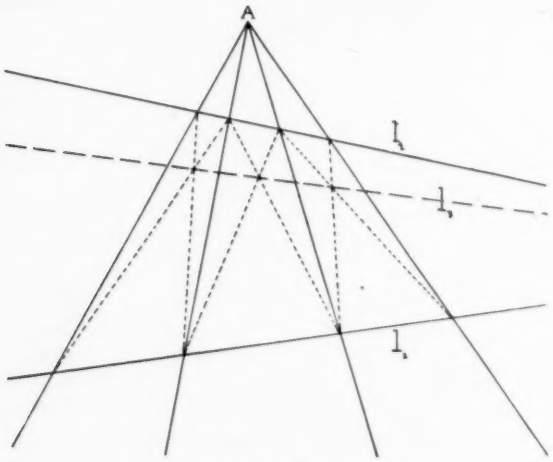


FIG. 3

From projective geometry comes also the following theorem which is of fundamental importance: Consider any four lines through a point $A$; let two new lines, $l_1$ and $l_2$, each cut these four lines in the points $a_1$, $a_2$, $a_3$, $a_4$, and $b_1$, $b_2$, $b_3$, $b_4$, respectively. You will, if you are not familiar with the result, find it a very puzzling problem to establish any relation between the lengths of the segments cut off on $l_1$ and those cut off on $l_2$. Such a relation is:

$$\frac{\dfrac{a_1 - a_2}{a_3 - a_2}}{\dfrac{a_1 - a_4}{a_3 - a_4}} = \frac{\dfrac{b_1 - b_2}{b_3 - b_2}}{\dfrac{b_1 - b_4}{b_3 - b_4}}.$$

It is expressed by saying that the cross-ratio of the four $a$-points is equal to the cross-ratio of the four $b$-points. Here, again, the mind is fascinated by the large degree of freedom left in choosing the lines $l_1$ and $l_2$, leading in all cases to the same value of the cross-ratio. In pure projective geometry, measurement, of lengths is not based on an ordinary scale of "equidistant"

points, but on more complicated scales, and the cross-ratio of four points plays in this measurement a basic rôle.

I believe these few examples, which could be multiplied indefinitely, will be sufficient to illustrate what I have in mind: Mathematics in itself possesses a marvelous beauty; nor is it necessary to be a professional mathematician to secure a feeling of this beauty; and, since cultural value must be ascribed to everything which enriches our life with new values of beauty, we are justified in including this aspect as a final argument for the cultural value of mathematics.

## HELP THE NATIONAL COUNCIL GROW!

The total number of subscribers has increased from 4,344 in January 1928 to 4,819 in January 1929—an increase of almost 400. It is hoped that teachers in those states where the membership is low or has actually decreased will do all they can to present the work of the Council at their various meetings. Pamphlets describing the publications of the Council will be sent upon request.

| | 1928 | 1929 | | 1928 | 1929 |
|---|---|---|---|---|---|
| Alabama | 76 | 51 | Nevada | 2 | 2 |
| Arizona | 10 | 12 | New Hampshire | 21 | 20 |
| Arkansas | 26 | 31 | New Jersey | 124 | 157 |
| California | 157 | 169 | New Mexico | 9 | 10 |
| Colorado | 57 | 65 | New York | 568 | 599 |
| Connecticut | 78 | 88 | North Carolina | 93 | 100 |
| Delaware | 3 | 3 | North Dakota | 12 | 8 |
| Florida | 32 | 33 | Ohio | 223 | 267 |
| Georgia | 38 | 27 | Oklahoma | 56 | 95 |
| Idaho | 14 | 15 | Oregon | 24 | 23 |
| Illinois | 340 | 357 | Pennsylvania | 374 | 401 |
| Indiana | 143 | 156 | Rhode Island | 24 | 35 |
| Iowa | 99 | 156 | South Carolina | 23 | 28 |
| Kansas | 178 | 184 | South Dakota | 20 | 19 |
| Kentucky | 34 | 29 | Tennessee | 29 | 41 |
| Louisiana | 42 | 35 | Texas | 161 | 146 |
| Maine | 30 | 33 | Utah | 5 | 10 |
| Maryland | 89 | 71 | Vermont | 17 | 18 |
| Massachusetts | 263 | 278 | Virginia | 49 | 63 |
| Michigan | 150 | 231 | Washington | 33 | 36 |
| Minnesota | 112 | 119 | West Virginia | 36 | 44 |
| Mississippi | 39 | 47 | Wisconsin | 125 | 127 |
| Missouri | 88 | 89 | Wyoming | 14 | 16 |
| Montana | 11 | 12 | Philippine Islands | 14 | 12 |
| Nebraska | 74 | 103 | Foreign Countries | 103 | 107 |

# CONTROVERSIES ON MATHEMATICS BETWEEN WALLIS, HOBBES, AND BARROW

By FflorIAN CAJORI

*University of California*

In the writings of John Wallis, Thomas Hobbes and Isaac Barrow, persons proficient in the dramatic art will find rich material for a mathematical tragedy or comedy. Hobbes, the aged philosopher superficially versed in all fields of human knowledge, presumes to show mere professors of mathematics that a philosopher from his wider outlook easily disposes of problems which had baffled mathematicians for ages—such problems as squaring the circle, trisection of an angle and the duplication of the cube. Wallis, young and ambitious, a highly trained mathematician and gifted also in the power of satire and ridicule, demolishes Hobbes' geometric structure, leaving no stone unturned.[1] Barrow, the theologian and mathematician, assuming the position of a superior judge, takes under advisement the various mathematical views advanced by Hobbes and Wallis, and finds something to praise and much to criticise in the philosophy of both. Hobbes the septuagenarian emerges from the conflict battered and torn. Wallis shows little outward evidence of the struggle, but inwardly repents for the mass of ridicule which he had heaped upon the aged philosopher; Wallis refuses to have his controversial articles against Hobbes included in his collected works. Such in crude outline is the material available for the dramatist. We note a few of the topics discussed by these distinguished seventeenth century thinkers.

## Foundations of Mathematics

John Wallis of Oxford founded mathematics on arithmetic; Barrow founded it on geometry. The very title of Wallis' most original book, the *Arithmetica infinitorum*, indicates his arithmetical procedure, even though the main subject of investigation

[1] A good account of the controversial writings between Wallis and Hobbes is found in the articles "Hobbes" and "Wallis" in the eleventh edition of the Encyclopædia Britannica.

in the book is the quadrature (*i.e.* determination of areas) of geometric figures. The work in which Wallis gives more explicit expression to his views on the philosophy of mathematics is entitled *Mathesis universalis: sive arithmeticum opus integrum*. With Wallis arithmetic was an abstract science to which geometry was subordinate. To Hobbes arithmetical and algebraic reasoning made little appeal, and he found fault with Wallis for excessive use of algebraic symbols. He insisted that Wallis "mistook the study of *symbols* for the study of *geometry*," [2] and referred to the "scab of symbols" in Wallis' geometry of the conic sections. On the relative place of arithmetic and geometry, Barrow also disagreed with Wallis, but on other grounds. The viewpoints of both may be presented most compactly by reproducing Barrow's quotations from Wallis' *Mathesis universalis*, and also Barrow's remarks on these quotations. The following extract from Wallis shows that he considered arithmetic the more general of the two sciences: "Simply because a line of two feet added to a line of two feet makes a line of four feet, it does not follow that two and two make four; on the contrary, the former follows from the latter." [3] To this Barrow answers: "Whence comes it to pass that a line of two feet added to a line of two palms cannot make a line of four feet, four palms, or four of any denomination, if it be abstractedly . . . true that $2+2$ makes 4? You will say, that these numbers are not applied to the same matter or measure; and so say I too; from whence I infer that $2+2$ makes 4, not from the abstract reason of the numbers, but from the condition of the matter to which they are applied." [4] Barrow starts from the consideration that only concrete numbers can be added, and those must be of the same denomination. From these concrete relations Barrow infers that abstractly 2 and 2 make 4. Wallis, on the other hand, begins with the concept of the numbers 1, 2, 3, . . . disassociated from any particular objects or units of measure; with him $2+2=4$ is an abstract relation which can be applied to special objects or units of the same kind.

[2] Thomas Hobbes, "Six Lessons to the Professors of the Mathematics," in *The English Works of Thomas Hobbes*, edited by W. Molesworth, Vol. VII, London, 1845, p. 187.

[3] John Wallis, *Mathesis universalis*, Oxford, 1657, p. 69.

[4] Isaac Barrow, Mathematical Lectures, transl. from the Latin by John Kirkby, London, 1734, p. 37.

Wallis' point of view is brought out somewhat more sharply to the foreground by the following quotation from his book:[5] "The assertion concerning the equality of the number 5 with the numbers 2 and 3 taken together, is a general assertion, which is applicable not only to geometry but to all other things; for also 2 angels and 3 angels are 5 angels." Barrow on the other hand maintains that "no particular number taken separately and absolutely can signify anything certain. . . . I note that numbers of themselves can neither be added to nor subtracted from one another, so as in one case to compose a sum or in the other case leave a difference."[6] "Mathematical number has no real existence proper to itself." After showing how the square root of 3 years can be found geometrically with the aid of a circle 4 yards in diameter. Barrow says that numbers like $\sqrt{3}$ "cannot even in thought itself be abstracted from all magnitude." While he would not go so far as to expunge arithmetic out of the list of mathematical sciences, he affirmed "the whole of mathematics to be in some sort contained and circumscribed within the bounds of geometry." The verdict of time has gone against the views of Barrow. Arithmetical analysis has been found a much keener and more powerful procedure than geometry. For example, the impossibility of the quadrature of the circle, the trisection of an angle and the duplication of a cube, by the use only of a pair of compasses and an unmarked ruler, has been established conclusively, not by reasoning involving geometric figures, but by reasoning involving arithmetical steps and the study of the properties of algebraic equations. Wallis anticipated some nineteenth century ideas on arithmetization. The most rigorous modern expositions of the theory of limits and of the differential and integral calculus rest on the modern number-system with its careful definition of irrational number.

### Geometrical Proportion

Hobbes declares that Euclid's definition of proportion is impossible of application, because it requires an infinite number of tests.[7] Four magnitudes $a$, $b$, $c$, and $d$ are in proportion, according to Euclid,[8] if simultaneously one or the other of the re-

[5] J. Wallis, *Mathesis universalis*, 1657, pp. 73, 226.

[6] I. Barrow, *op. cit.*, pp. 29, 35, 41, 45.

[7] Th. Hobbes, *op. cit.*, Vol. VII, p. 243.

[8] Euclid's *Elements*, Book V, Definition 5.

lationships, $ma \gtreqless nb$, $mc \gtreqless nd$, holds for any positive integral values of $m$ and $n$. That is, if for a pair of values of $m$ and $n$, we have $ma > nb$, then we must have also $mc > nd$, otherwise $a$, $b$, $c$, and $d$ could not be proportional. But we are not certain that $a$, $b$, $c$, and $d$ are proportional, unless we know that these inequalities or equalities hold simultaneously for *all* possible pairs of value of $m$ and $n$. Hobbes insists that multiplication by all numbers $m$ and $n$ is impossible, for there are infinitely many of them. Indeed, since the span of our lives is *finite*, it is not possible to carry out separately each of these trials. Does not Hobbes' criticism of Euclid appear altogether valid? The nature of this criticism is, by the way, the same as that levelled by certain modern mathematical philosophers against other workers in the field of transfinite numbers. Nevertheless, in spite of Hobbes' criticism, Euclid's definition has been widely admired. It has been declared to be equivalent to the "Dedekind cut," famous in the theory of irrational number.[9] But to return to the seventeenth century, Euclid's definition was staunchly defended by Barrow. He says truly:[10] "This simultaneous Defect, Excess or Equality may in some cases happen to quantities not proportional, but it happens to proportionals alone universally." Barrow proceeds to explain, by example, how a single test may suffice, for the same reason that proving that one ray of light travelling parallel to the principal axis of a parabolic mirror is reflected through the focus, proves that all such rays are reflected through the focus. The reduction of an infinite number of proofs to a single proof results from physical and geometric properties applicable to all rays. Barrow refers to Euclid's proof that two triangles having the same altitude have their areas proportional to their bases, even when the bases are incommensurable.[11] Barrow is too verbose for quotation, but he shows that the infinity of trials are in this instance reduced to one trial because of the theorem previously established in Euclid that, in two triangles, having the same height, the area of the first triangle is equal to or greater than the area of the second, according as the base of the first is equal to or greater than the base of the second.

[9] J. W. Young, *Fundamental Concepts of Algebra and Geometry*, New York, 1911, pp. 103–106.

[10] Barrow, *op cit.*, p. 393.

[11] Euclid's *Elements*, Book VI, Prop. I.

### Hobbes' Claim that Algebra and Geometry may Yield Contradictory Results

If $a$ is a number, the interpretations of $a$, $a^2$, $a^3$, as "line," "square," "cube," respectively, caused Hobbes much trouble. So did $\sqrt{a}$ as the side of a square and $\sqrt[3]{a}$ as the side of a cube. He seriously objected to $\sqrt{8} = (\sqrt{2})^3$, for "here the root of 8 is put for the cube of the root of 2. Can a line be equal to a cube?"[12] Considering $9\sqrt{2} = \sqrt{162}$ he says, "How does 9 roots of 2 make the root of 162?", for 9 multiplied into $\sqrt{2}$ gives a rectangle, but $\sqrt{162}$ is a line. "I see the calculation in numbers is right, though false in lines. The reason whereof can be no other than some difference between multiplying numbers into lines or planes, and multiplying lines into the same lines or planes." He was very realistic (should say materialistic?) in assuming a point and a line as having some thickness; "every one line has some latitude." As Wallis expressed it:[13] "Since it is, with Hobs, not conceivable for anything *to be*, which is not *Body*, it must be as impossible, for a Point to be if it be not *Great*." Hobbes' assumption established a cleavage between arithmetical and geometrical processes. He had given a construction effecting, as he thought, the duplication of the cube.[14] A French critic pointed out Hobbes' error by assuming suitable arithmetical values and proceeding arithmetically. There arose a discrepancy equal to the difference between $\sqrt{1682}$ and $\sqrt{1681}$. Hobbes claims that his "demonstration is not confuted; for the point $Y$ will have latitude enough to take in that little difference which is between the root of 1681 and the root of 1682." His geometry, he tells us, is not refuted by any but mathematicians, whose judgment in this case is not to be credited. He felt that he had "wrested out of the hands of [his] antagonists this weapon of algebra, so as they can never make use of it again." Barrow did not go to such extremes: "Be it far from me to take away or seclude a science so excellent and profitable as that of numbers from the Mathematics. I will rather restore it into its lawful place, as being removed out of its proper seat, and ingraff and unite it again into its native

[12] Th. Hobbes, *op cit.*, Vol. VII, p. 66.

[13] Hobbius, *Heauton-timorumenos, or A Consideration of Mr. Hobbes his Dialogues*, by John Wallis, Oxford, 1662, p. 31.

[14] Th. Hobbes, *op. cit.*, Vol. VII, p. 60.

geometry, the stock from whence it has been plucked (p. 29)." Hobbes thought he saw an impassable chasm between algebra and geometry. He had lost sight of the advantages resulting from the abstraction made by the Greeks which considers points and lines as having no thickness. In 1904 Horace Lamb, addressing Section *A* of the British Association, said: "If any scientific invention can claim pre-eminence over all others, I should be inclined myself to erect a monument to the unknown inventor of the mathematical point, as the supreme type of the process of abstraction which has been a necessary condition of scientific work from the very beginning."

The reader will find much in Wallis, Hobbes, and Barrow, touching the philosophy of mathematics, which we omit for lack of space, such as the definition of parallel lines, the recognition of irrationals as numbers, the definition of an angle, the angle of contact, the possibility of applying the method of superposition to solid bodies, change of magnitude with change of place, the existence of indivisible lines, whether magnitude be peculiar to bodies or belong to spirits also, the cycloid, and infinity.

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If the National Council of Teachers of Mathematics is to really assist the classroom teachers of mathematics in America, we shall have to get more of them interested in the work we are trying to do. We are accomplishing a great deal at the present, but if we are to develop properly the *group consciousness, group pride*, and *group enthusiasm* which is so fundamental to success two things at least must be done. In the first place every teacher of mathematics should be a member of the Council and should read the MATHEMATICS TEACHER regularly. Secondly the teachers should read the Yearbooks faithfully so as to keep abreast of the times. We earnestly request the cooperation of every individual member of the Council in our attempt to bring this about. See the description of the Fourth Yearbook on page 188.

# THE NEWER TYPE OF MATHEMATICS [1]

By JOHN A. SWENSON

*Wadleigh High School, New York City*

The Report of the National Committee on Mathematical Requirements appeared in 1922. This report makes recommendations which when adopted will bring us into closer agreement with the mathematical curricula of the other leading countries of the world. What these recommendations are need not be stated here since they are familiar to most teachers of mathematics.

The revised syllabus of the College Entrance Board, issued in 1924, follows to a great extent the recommendations made by the National Committee. But this syllabus is fundamentally an examination syllabus intended primarily for a finished product, and hence need not concern itself with the order and the combinations in which the various parts of high school mathematics should be taught. The tentative Regents' syllabus in elementary algebra which went into effect this September follows the recommendations made by the National Committee in regard to algebra and numerical trigonometry, but makes no provision for demonstrative geometry in the ninth school year. The junior high school syllabus of the same date for the seventh, eighth, and ninth school years provides for a very limited amount of demonstrative geometry in the ninth year. Hence we have two conflicting syllabuses intended for pupils that must finally be brought together in the tenth year of the senior high school.

The Regents' syllabus committee no doubt considers demonstrative geometry a desirable thing for the ninth school year, but was too timid to recommend its inclusion in the syllabus for that part of the ninth year population which is housed in the senior high school. Why this timidity about demonstrative geometry in the ninth year? On page 31 of its report, where schools organized on the 8–4 plan are dealt with, the National Committee says: "Under these conditions it would seem desirable that the work of the ninth grade should contain both algebra and geometry. It

[1] Reprinted from *Bulletin of High Points* in the Work of the High Schools of New York City for December, 1928.

is, therefore, recommended that about two-thirds of the time be devoted to the most useful parts of algebra, including the work on numerical trigonometry, and that about one-third of the time be devoted to geometry, including the necessary informal introduction and, if feasible, the first part of demonstrative geometry."

In spite of these very plainly worded recommendations, many authors of old-fashioned algebras with a few pages devoted to numerical trigonometry, but no demonstrative geometry whatsoever, have boldly announced in the preface: "This book follows the recommendations of the National Committee for the ninth year."

The recommendation of the National Committee in regard to demonstrative geometry in the first year of the senior high school has been tried in the Wadleigh High School for a number of years and found to work very satisfactorily. Obviously such pupils do better in their Regents' examination in geometry than those who begin geometry in the tenth year. But this is not the end of the story. They also do better work in intermediate algebra. In fact, some of the brighter ones have been able to omit intermediate algebra and take advanced algebra immediately after plane geometry. This is made possible by the fact that logarithms and graphs, both equation and statistical, are taught in connection with elementary algebra.

When no Regents' examination is given at the end of the ninth year, more time may be given to the development of concepts in mathematics and less to manipulations. This is a good deal better for the mathematical future of the youngster than the extreme attention some teachers give to mechanical manipulations and cramming for examinations. Schools which devote themselves too exclusively to preparing for the Regents' examination in elementary algebra, often have a hard time to finish the present syllabus in intermediate algebra.

In demonstrative geometry the pupil meets his first chance for easy logical reasoning and, for this reason, he should be introduced to it as early as possible. If some geometry is given in the ninth year, there will be time to keep elementary algebra alive during the tenth year. This means using algebra and geometry together, which is a very good thing for each subject. Furthermore, the time over which a subject is spread is very important.

It is, after all, the frequent coming back to a topic and meeting it in many different settings that makes it our permanent property and available for applications to practical situations when needed. Practically all the secondary schools of Europe teach their algebra and geometry together and spread them over a period of at least three years.

Also, if demonstrative geometry is begun in the ninth year, it is comparatively simple to classify the pupils according to ability in geometry in the tenth year, if such classification is desired. Hence, it seems to me that a course in general mathematics is desirable for the ninth and tenth years. But it should not end there. Mathematics should be cumulative and aim to use everything in the subject of mathematics that has gone before. Intermediate algebra and the higher branches should continually use the subject matter of geometry and even enlarge on it, making use of more advanced methods made possible by the greater maturity of the student. Algebraic methods in geometry is a far better topic for intermediate algebra than many of the puzzle problems in algebra now so generally given.

Throughout mathematics more attention should be given to general methods, since such methods will give a student more power and a better insight into what mathematics really is than a great variety of special methods. Such powerful tools and methods as differentiation and integration should be introduced in the twelfth year instead of the obsolete methods now used in advanced algebra and solid geometry. In accordance with the recommendations of the National Committee, calculus has been used in connection with advanced algebra, trigonometry, and solid geometry in Wadleigh for a number of years. These three old-time subjects have their usual portion of the program, but they have been related and unified by means of the calculus. But the contention of the National Committee that "calculus necessarily includes sufficient algebra and geometry to compensate for whatever diversion of time from these subjects is necessary" has been completely justified. The power gained through the introduction of the calculus has enabled the students to pass the old-time Regents' examinations in the three subjects referred to.

In the Third Yearbook of the National Council of Teachers of Mathematics, Professor Nordgaard of St. Olaf College, North-

field, Minnesota, reviews the various changes that have taken place in the teaching of college mathematics, and under the heading of "Reasons for the Change," says:

"There is no mistaking, however, which way the wind blows. The calculus is moving downwards."

"The most patent, though not the most fundamental, reason for the change is the age in which we live. Thanks to our generous system of high schools and modern means of communication the intellectual interests of the rank and file have never been so numerous nor has their participation in the conduct of the world's affairs ever been so active as at the present time. Ours is a scientific age, and science speaks through mathematics, notably through the calculus. Not only do physics, mechanics, and engineering depend on the calculus—everybody knows that—but radioactivity, medicine, and actuarial science frequently state the results of their latest researches in calculus form. Even insurance, statistics, heredity, and education speak the language of the calculus. In these days of ocean flights the aviation journals frequently contain articles that can not be understood without at least some knowledge of this subject. To know some calculus has become a necessity for an educated person."

Those of our educators who continually work against mathematics should keep in mind that with mathematics, as with any branch of learning, its practical and cultural value must be measured not by whether one can get along without it if one does not know it, but by whether and how much of it one will use if one does know it.

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As we go to press we are glad to announce a most successful program at the meeting of the National Council of Teachers of Mathematics at Cleveland on February 22d and 23d. The April number of the MATHEMATICS TEACHER will contain most of the papers read at the meeting.

# A STUDY OF PROCEDURES USED IN THE DETERMINATION OF OBJECTIVES IN THE TEACHING OF MATHEMATICS

By J. S. GEORGES

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The question of the general and specific functions of mathematics in secondary education has been a source of much discussion and dispute for many years. The curriculum studies which have been concerned chiefly with attempts to determine educational objectives have used such methods as analysis of pupil activities, analysis of social needs, opinions of competent persons and authorities in the field, and analysis of existing objectives found in educational investigations, courses of study, and textbooks. The outcome of the specific influence of these studies has been the formulation of a great number of courses of studies with lists of general course-objectives and specific unit-objectives. While it is true that in many cases the teachers of mathematics, who should be the most vitally interested in the determination and evaluation of the objectives relating to their subjects have been consulted in the matter, nevertheless, in general the solution of the problem has been left to the "experts" and more "qualified" persons; the teachers have accepted the lists of objectives presented in the courses of study and have tried to attain them the best way they could. It is felt that as a point of pedagogical interest it is worth while to note the reactions of teachers of mathematics toward the proposed lists of objectives and the procedures followed in their determination. This study reports the evaluation by a group of teachers of secondary school mathematics of the merits of certain procedures used in the determination of mathematical objectives.

The study was conducted at the University of Wisconsin during the summer session of 1928. A list of six different procedures was submitted to the writer's class in "Investigations in the Teaching of Mathematics," out of which each member of the class was to select the "best" and the "least" efficient procedure. Provision was made in the test for recommending a procedure either different from the six submitted or a combination of any number of those submitted, if the student felt that such a procedure would be more efficient. In each case definite

reasons were to be given why a particular procedure was selected as the best or the least efficient. The results were then tabulated and submitted to the class for further study. The test is reproduced below.

"Suppose you are on a committee appointed by the Board of Education to formulate certain objectives for the course in mathematics which you are teaching. On the basis of these objectives a reorganization of the course is to be made. The following are some of the procedures employed:

1. The bankers, merchants, professional men, etc., of the community are consulted and their opinions are used in determining the objectives for the course.
2. The pupil-activity of those taking the course is observed, recorded, and analyzed, and the pupils' interests in the subject, as well as their dislikes for the subject, are made the basis for the formulation of the objectives.
3. The objectives recommended by authorities in the mathematical world, such as the National Committee on Mathematical Requirements, are accepted.
4. The teachers of mathematics on the staff of the school submit lists of objectives from which the committee selects the desired objectives.
5. The committee studies the published works of those investigators in the field of education who have made a study of the problem, and follows their conclusions and recommendations.
6. The courses of study of other school systems are consulted and a list of objectives is selected from them.

Complete the following statements:

1. I consider procedure No.....the best.
2. I consider procedure No.....the least efficient.
3. I consider the following procedure more efficient than any of those given.
4. My reasons for selecting the above procedures are:"

The 42 papers which furnish the data for this study are those of students who were either teachers of mathematics or supervisors of instruction. Table I, which presents data concerning their qualifications, shows that we may accept this group as a representative group of teachers of secondary school mathematics with average training and experience.

Table II gives the frequency distribution for the submitted list of six procedures. The index numbers in column four are found by subtracting the number in the "least efficient" column from the number in the "best" column and dividing this difference by the total; the ratios are then changed to per cents. The index numbers show that procedure No. 3 is considered the best, with procedure No. 5 a close second; and pro-

TABLE I

DATA CONCERNING EDUCATION AND EXPERIENCE OF THE GROUP

| | Number | Per cent. |
|---|---|---|
| 1. Academic training: | | |
| No degrees; two or more years of college work | 3 | 7.1 |
| Bachelor's degree | 32 | 76.2 |
| (Candidates for Master's degree) | (21) | |
| Master's degree | 7 | 16.7 |
| (Candidates for Doctor's degree) | (5) | |
| Total | 42 | 100.0 |
| 2. Teaching experience: | | |
| No experience | 1 | 2.4 |
| 1–5 years | 16 | 38.1 |
| 6–10 years | 12 | 28.6 |
| 11–15 years | 10 | 23.8 |
| 16–20 years | 3 | 7.1 |
| Total | 42 | 100.0 |
| Average experience—7.8 years | | |
| 3. Type of positions held: | | |
| Teacher, junior high school | 12 | 28.6 |
| Teacher, senior high school | 22 | 52.4 |
| Teacher, college | 2 | 4.8 |
| Supervisor | 2 | 4.8 |
| Principal | 3 | 7.1 |
| Superintendent | 1 | 2.4 |
| Total | 42 | 100.1 |

cedure No. 1 is the least efficient, followed closely by procedure No. 6. However, since about two-thirds of the group recommended other procedures, these results are to be interpreted in the light of the recommended procedures.

TABLE II

CLASSIFICATION OF SUBMITTED PROCEDURES ACCORDING TO FREQUENCIES

| Procedure | Frequency Best | Frequency Least Efficient | Index $r = 100(b - l/t)$ |
|---|---|---|---|
| 1 | 1 | 26 | — 59.5 |
| 2 | 5 | 5 | 0. |
| 3 | 16 | 2 | 33.3 |
| 4 | 6 | 2 | 9.5 |
| 5 | 14 | 2 | 28.6 |
| 6 | 0 | 5 | — 11.9 |
| Total | 42 | 42 | 0. |

There were no new procedures recommended. Table III gives the classified list of procedures recommended by 69 per cent of the group, and Table IV presents the frequency distribution of the recommended procedures. The index numbers in column 3 of Table IV are found as follows: The sum of the index numbers of the procedures entering a combination from Table II is multiplied by the frequency of that combination, and the result is divided by the total. This gives proper weight to the procedures entering into a combination of any proposed procedure in view of their relative standing in Table II.

TABLE III

LIST OF RECOMMENDED PROCEDURES

*a*. A combination of procedures 1, 2, 3, 4, 5, and 6.
*b*. A combination of procedures 1, 2, and 4.
*c*. A combination of procedures 1, 4, and 5.
*d*. A combination of procedures 2 and 3.
*e*. A combination of procedures 2, 3, and 4.
*f*. A combination of procedures 2, 3, and 5.
*g*. A combination of procedures 2, 3, 5, and 6.
*h*. A combination of procedures 2 and 6.
*i*. A combination of procedures 3 and 4.
*j*. A combination of procedures 3, 4, and 5.
*k*. A combination of procedures 3, 4, and 6.
*l*. A combination of procedures 3 and 5.
*m*. A combination of procedures 3, 5, and 6.

TABLE IV

CLASSIFICATION OF RECOMMENDED PROCEDURES ACCORDING TO FREQUENCY

| Procedure | Frequency | Index $I = f\Sigma r_i/t$ |
|---|---|---|
| *a* | 6 | 0 |
| *b* | 1 | −1.7 |
| *c* | 1 | −0.7 |
| *d* | 2 | +2.3 |
| *e* | 1 | +1.4 |
| *f* | 1 | +2.1 |
| *g* | 3 | +5.2 |
| *h* | 1 | −0.4 |
| *i* | 1 | +1.4 |
| *j* | 1 | +2.5 |
| *k* | 2 | +2.1 |
| *l* | 7 | +14.9 |
| *m* | 2 | +3.4 |
| Total | 29 | |

The results of Table IV definitely show that procedure *l*, which is a combination of submitted procedures 3 and 5, is the best procedure to be used according to the judgment of 69 per cent of the group. Recalling that procedure No. 3 represents the acceptance of the objectives recommended by the National Committee, and procedure No. 5 follows the conclusions and recommendations of investigators in the field of objectives, the best procedure recommended is a combination of the two, or rather a modification of the one in view of the other. Of the 13 members of the group who did not propose any new procedures, 4 selected procedure No. 3 and 5 selected procedure No. 5 as the best. Combining this with the fact that procedures Nos. 3 and 5 stand first and second respectively, according to the judgment of the whole group, we conclude that procedure *l* receives the sanction of the group as a whole, that is, the best procedure to be used in determining objectives for courses in secondary school mathematics is to follow the recommendations of the National Committee, modifying the list in view of the findings of investigators in the field to meet the local demands of the school.

The reasons given by the members of the group for or against the use of a procedure, either alone or in combination with other procedures, were classified according to the procedures, and are presented here, because of their significance in showing to what an extent the judgment of this group of teachers may be accepted as reliable. They reflect in an interesting way the opinions expressed by Tables II and IV.

REASONS FOR SELECTING A PROCEDURE OR FOR ITS USE IN CONNECTION WITH OTHER PROCEDURES

*Procedure 1.*

1. Would bring in the practical point of view.
2. Would eliminate useless contents.

*Procedure 2.*

1. It is an experimental type of procedure.
2. It is a scientific method.
3. It is the natural and logical way.
4. Objectives would be efficient and practical—would meet the needs of the pupils.
5. Objectives differ in different communities.
6. Would eliminate the objectives beyond the pupil's ability.
7. Cannot put the subject across without taking account of the pupil's interest.
8. Objectives based upon the interests of pupils can be readily realized (attained).

9. Pupil is the most important person in the world today.

*Procedure 3.*

*A.* The personnel of the committee.

1. Committee made up of authorities in field.
2. Committee made up of men of experience.
3. Men who are the cream of the mathematical world.
4. Membership included representatives of the various fields—experts in mathematics, teachers of mathematics, investigators in education, administrators.
5. Committee included the best representatives the country over.
6. Committee was in position to carry on extensive investigations.

*B.* The technique used.

1. Have made a careful study of the problem.
2. Have covered the entire field.
3. Have considered the needs of the children.
4. Study includes every activity of life.
5. Considered the problem from the viewpoint of the teacher as well as the pupil.
6. Considered social, vocational, as well as the intellectual phases.
7. Included the opinions and work of investigators in the field.
8. Consulted business men.
9. Carried on experiments and investigations on various problems.
10. They used the best methods available.
11. Work extended over a long period of time.
12. Work carried on in an orderly manner.
13. Studied analytically the values of different methods, organizations, and objectives.
14. Study acknowledged as an efficient piece of work.

*C.* The list of objectives presented.

1. Results applicable to all schools.
2. List of objectives is complete.
3. Results are reliable, based on scientific investigation.
4. Objectives are correct, based on best methods of teaching.
5. List represents the broader point of view.
6. Objectives have been determined by experience.
7. Objectives have guarded the mathematical side of the question.
8. Results are presented in readable form.

*Procedure 4.*

*A.* The training of teachers.

1. Are familiar with the field of mathematics.
2. Know the possibilities of correlating mathematics with other subjects.
3. Know the requirements for college entrance.

*B.* The experience of teachers.

1. Have opportunity to study pupil activity.
2. Know the needs of their own school.
3. Know what findings of investigators are better suited for their schools.
4. Could decide on the teachability of objectives.
5. Have knowledge of immediate needs.
6. Could use other procedures and determine workable objectives.
7. From experience would eliminate the undesirable objectives.
8. Would consider objectives from all angles.

*C.* The interest of teachers.

1. They are most vitally interested in the problem.
2. They would make a careful study of the investigations of others.
3. They are unprejudiced.

*Procedure 5.*

*A.* The investigators.

1. Have studied the problem carefully.
2. Have considered all the elements of the problem.
3. Are interested in the problem.

*B.* The technique.

1. More expert procedure used.
2. A combination of all other procedures.
3. Probably based upon sound educational philosophy.
4. Results more reliable because proved experimentally.
5. Objectives based on practicability as working in various school systems.
6. Results of a general character, applicable to all sections of the country.
7. Results based upon a scientific method of research.
8. Results based upon experimentation.

*Procedure 6.*

1. Would standardize the scope of material.
2. Cities like Fresno, St. Louis, Denver, that have spent much money and employed experts to set up proper objectives.
3. Might include the objectives of those schools which are of a higher standing than own school.
4. Something gained by investigating the work done by other schools.
5. Profit by the experience of other schools.

REASONS AGAINST SELECTING A GIVEN PROCEDURE, OR USING IT IN CONNECTION WITH OTHER PROCEDURES

*Procedure 1.*

*A.* Bankers, merchants, professional men, are not capable to discuss or solve pedagogical problems.

1. Do not know technique of teaching to formulate workable objectives.
2. Have little or no knowledge of sound pedagogical principles.

3. Professional knowledge of little value in organization of school subjects.
4. Have no knowledge of the teachability of the material they would suggest to be included.
5. Have no knowledge of child needs (at that age).
6. Would not consider the teacher's viewpoint.
7. Would not consider the learner's viewpoint.
8. Have no idea of end to achieve in mathematics.
9. Unable to see
    *a*. Disciplinary values of mathematics.
    *b*. Cultural values of mathematics.
    *c*. Recreational values of mathematics.
    *d*. Social values of mathematics.
    *e*. Appreciation of mathematical concepts.
    *f*. Formation of good mathematical habits.

*B*. Bankers, merchants, professional men do not have an adequate mathematical training which is necessary for a comprehension of the true objectives of mathematical instruction.
1. Know little or nothing of algebra and geometry.
2. They confess that their knowledge of mathematics is very limited (next to nothing).

*C*. Because of their special interests, bankers, merchants, and professional men would have a prejudiced viewpoint.
1. Recommendations based only on personal opinion.
2. Opinion of business men not reliable.
3. Objectives would be only vocational, they see only the practical side.
4. Objectives would be based upon the use of mathematics in business or profession, and that is very limited.
5. Point of view would be decidedly warped.
6. Would have a cramped idea of the problem.
7. Their opinions fluctuate with every tide.
8. Opinions based upon quick judgments.

*D*. Objectives proposed by bankers, merchants and professional men would not be sound educationally.
1. They are not interested in the field.
2. They have given little previous thought to the problem.
3. The man about town will tell more about running a school than any trained or experienced educator.
4. Everybody's business is nobody's business.
5. They are too busy and would not give sufficient time to the study of the problem.
6. They are not in position to make special studies in education.
7. Objectives would be only of a local nature.
8. Objectives other than just the community needs should be considered.
9. Mathematics would be curtailed.
10. Not enough objectives—could teach the whole course in three or four weeks.

*Procedure 2.*

1. Too many elements involved—type of pupils, method of presentation, personality of teacher, etc.
2. Likes and dislikes depend upon the course, hence the resulting objectives would depend upon the course.
3. Pupils' likes and dislikes have nothing to do with objectives.
4. Personal likes and dislikes often determined by unimportant factors.
5. Children are prejudiced against mathematics.
6. Interest varies from year to year.
7. Few of the aims ever realized in a class.
8. Method is not scientific.

*Procedure 3.*

1. Objectives too theoretical.
2. Objectives based upon opinion.
3. May not meet the local needs and requirements.
4. Not applicable to every section of the country.
5. Results are old now, should not be taken as a whole in every community.
6. Recommendations should be tested out by local experiments.
7. Objectives should be supplemented from other sources.

*Procedure 4.*

*A.* The teachers.

1. Not capable to decide.
2. Their experience in research and observation very limited.
3. Have limited knowledge of the problem.
4. Not capable to consider all objectives.
5. Would select objectives without knowing whether good or not.
6. Would give answers on the spur of the moment.
7. They lack perspective in the general objectives of education.
8. Inadequate experimental background.

*B.* The method.

1. Highly subjective.
2. Based upon local conditions.

*C.* The objectives.

1. Would be second hand.
2. Would have no verification back of them.

*Procedure 5.*

1. Lacks elements of locality.
2. Not broad enough in scope.
3. Not primarily concerned with mathematics.
4. Many educational investigators not mathematicians.
5. Many educational investigators not experienced teachers of mathematics.

*Procedure 6.*

1. Danger in "copy work."
2. Not sure how functioning.
3. No knowledge of how objectives were selected.
4. May not prove practical in local schools.

*Conclusions:* Bearing in mind that the opinions of a very small group of teachers, no matter how representative, can at the best only reflect in a very limited way the reactions of teachers of mathematics toward any educational policy, the results of this study are to be interpreted not as a critical analysis of the merits of any particular procedure, but rather as indicative of the extent of teachers' interest in one of the fundamental problems in the teaching of mathematics. We are told that a list of objectives "should not be imposed upon the schools by some higher authority; on the contrary it should be the result of much thought and discussion on the part of those who will actually use it, who know the work of the class room, and who themselves are the authority as to what can be expected of children."[1] We ask the reader, who may be a teacher or investigator or both, whether this group of teachers have or have not shown themselves qualified, by their judgments as herein reported, to solve the problem of the determination of objectives satisfactorily. The following points of interest are to be noted:

1. That the report of the National Committee on Mathematical Requirements has had great influence upon research and investigation in the problem of the determination of the objectives of mathematics; that the recommendations of the Committee are to be followed closely, taking into consideration such modifications as are necessary to meet the local needs; and that the work of the Committee is considered as of outstanding importance by the teachers of mathematics.

2. That the work done by such investigators as Schorling, Reeve, and others, has had definite influence in the formulation of courses of study; that the findings of these men, who are also teachers of mathematics, are accepted by the teachers of secondary school mathematics as of greater significance than those of educators whose primary interests are not mathematical in nature.

3. That such studies as Wilson's employing the opinions of business and professional men in the determination of mathematical objectives are not looked upon with favor by the teachers of secondary school mathematics.

4. That pupil activity analysis, so much stressed by some educators, is not a dependable procedure to be used alone in the determination of mathematical objectives.

5. That most school systems may find it profitable to accept the lists of objectives determined by investigators in the field and modify the list in view of the recommendations of the National Committee to meet their local demands.

[1] Smith, David Eugene and Reeve, W. D. *The Teaching of Junior High School Mathematics,* Ginn and Company, 1927, p. 1.

# THE USE OF MATHEMATICAL HISTORY IN TEACHING *

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For several centuries subject matter was considered the only thing of importance in a course in mathematics. Then came a demand for the teaching of the applications of mathematics. Consequently there was written into our textbooks various types of applied problems. In the old order of things it was considered sufficient for the pupil to prove as an exercise-theorem that if the diagonals of a parallelogram are equal the figure is a rectangle. Now we suggest in connection with this exercise that a boy may test his accuracy in laying out a rectangular tennis court by measuring the diagonals. In more recent years we have also come to the realization that there is educational value in knowing the history of the subject; consequently we tell the pupil how and when the human race first discovered and proved this important fact about parallelograms with equal diagonals—or any other important and useful part of mathematics. So, in the last quarter of a century, we find the history of mathematics creeping into our high school textbooks. Let us consider then what value may be found in the history of mathematics, either in high school or college teaching, and how we may make use of mathematical history in teaching.

*A knowledge of the history of the development of the mathematical processes he is learning will kindle the pupil's interest in the subject matter.*

We hear much about human interest, the personal touch, and similar sentiments. Any newspaper publisher knows that matters pertaining to people have a greater news value than matters pertaining to things. He increases the circulation of his paper by printing what he knows the people like to read, and most of the people like best to read about what other folks have done or are doing. A reporter's orders are to get the names and facts about the people involved in a story, with pictures if possible.

* Read before the Kansas Mathematics Teachers Association at Topeka on February 2, 1929.

A plain story about an automobile wreck would attract only momentary interest if told without the names of the people concerned in it. Whether we know the victims of the accident or not, there is always some item concerning their lives or personalities that gives human interest to the news. So we pause in our reading because of the human interest and the incident is indelibly stamped upon our memory. I am sure that we all agree that the touch of human personality is the greatest factor in making news interesting.

We all know that efficient learning goes hand in hand with interest in the subject. Therefore, in seeking schemes to motivate the learning of fractions, for example, why not remove some of the abstractness from the processes by connecting them with human beings and let the pupil see how folks have struggled through greater difficulties than his to develop and simplify the rules he is learning? Would it not add to a seventh grade pupil's interest in fractions to be told something like the following?

Long, long ago when people could count only with the help of their fingers and toes, they had no need for fractions. Their herds could be counted in whole numbers. Their war clubs could be counted in whole numbers. Such an idea as half a tent or half a tree did not occur to them. As they became more civilized they tilled the fields, raised grain, and in various ways dealt with smaller things. Then the idea of measuring things gradually entered their minds. Corn and wheat came to be measured by bushels (or some similar measure) and soon they had half bushels and other fractional parts. It was thus that the idea of a fraction probably originated. The invention and use of fractions marked a very important step forward in civilization. Because of their invention in an arbitrary manner they are sometimes called artificial numbers, the whole numbers being called natural numbers.

Let us imagine a schoolboy of ancient Babylonia, in southern Asia, as he studied arithmetic more than 6000 years ago. With a hard, blunt-pointed instrument he made peculiar-looking wedge-shaped symbols on a soft, smooth clay surface. He could erase them with a flat board by patting the surface smooth again. His textbook, if he had any, had been written the same way on a sort of clay brick and had been baked hard so that the symbols could be erased. You wonder whether he had to learn fractions? He did, and to you his fractions may seem clumsy. All his fractions were 60ths, 3600ths, 216000ths, and so on, all the denominators being multiples of 60, just as our decimals are based on 10ths, 100ths, and 1000ths, and so on, all the denominators being multiples of 10. He wrote them much as you write decimals, omitting the denominators.

About the same time that the Babylonian youth was dealing with his 60ths, some schoolboy down in Egypt was struggling to express all his frac-

tions with 1's for numerators. No one there could conceive of a fractional part other than the unit fraction. The Egyptian lad was taught to write 2/3 as $(1/2 + 1/6)$, 7/8 as $(1/2 + 1/4 + 1/8)$.

A few thousand years later we find a Roman schoolboy basing his fractions on 12ths, or with multiples of 12 as denominators, as the Babylonian used 60ths.

Out of the centuries of effort has grown our present system by means of which we deal easily with problems in business and common life that would have been very difficult, or quite impossible, for even the most learned scholars of those ancient times.

Suppose a pupil is trying to master the rules of operation with what seems to him to be very abstruse algebraic symbols. It ought to awaken his interest to some degree to be told, for example, that the symbols as he is using them were scarcely known and used by very few when Columbus sailed for America; that an Alexandrian Greek, Diophantus, living in the latter part of the third century, was the first person known to use any systematic form of abbreviation in mathematics; that his abbreviations were very crude and limited; that he used a symbol not unlike an inverted *h*, **h**, for the unknown in an equation; that Vieta, a Frenchman living in the sixteenth century, was the first to use a complete system of symbols in algebra as we do; that he used the vowels, A, E, I, O, U for unknowns; that he was a rich man that could pay to have his work published, thereby causing the spread of his methods; that Descartes, the man who introduced coordinates into mathematics, a generation later originated the custom of using x, y, and z for unknowns; that Robert Recorde, an Englishman (1557), first used the sign, =, for "is equal to" and explained that he did it to save time, adding that no two things could be more equal than a pair of parallels; that *p* and *m* were used for a long time for plus and minus; that students in American universities less than 200 years ago wrote *aaaaa* instead of $a^5$ though Descartes had used the exponential symbol a century before; that several methods were used to point off decimals before our present one became popular, people once writing $7\underline{|465}$ to mean 7.465; that we have no internationally uniform method of pointing off decimals yet, the English using the decimal point higher than we do (7·465) and some Europeans the comma (7,465).

Besides seeing the personal touch of human beings in the subject matter he studies, the pupil should react in a wholesome

manner to the mere information about the origin of the things he is trying to learn. Thus they lose some of their abstractness. They become more concrete and real.

*A knowledge of the history of mathematics on the part of the teacher gives him a source upon which he may draw to enrich and enliven his teaching.*

There should be intervals of recreation in all labor. In periods of sustained and concentrated mental effort it is restful to have a recess in the nature of a change of thought. The successful public speaker realizes this and his speech is punctuated with jokes or stories. He does not allow his audience to grow tired or dull by too much concentration on a weighty subject.

In a mathematics class there is usually a sufficient variety of material in problem solving, but problem solving day after day grows monotonous. Some teachers enrich their teaching by illustrations of the various uses of mathematics. In addition to this every teacher can and should have stored in his mind ready for use the stories of the great mathematicians. In a few sentences the class can be told when studying similar triangles how Thales went down into Egypt and astonished the king by measuring the height of the pyramids without touching them. Many interesting stories of Pythagoras can easily be found. When a geometry class first learns to construct a perpendicular to a line they can be told that Oenopides, a Greek astronomer, was the first to do it, and he called it a gnomon-wise line after the gnomon that he and other astronomers would set up in a vertical position to cast a shadow for the purpose of determining the altitude of the sun. He *needed* to know how to set up a perpendicular and out of the necessity grew the solution of the problem.

What better impression could be made on a beginning class in geometry, wondering what it is all about, than to tell them something like this at the first meeting?

The sources of geometry have been in the world from the beginning. Little by little, through the ages, man has discovered geometric relations as he has needed them. When the earth was formed its shape was made nearly like a sphere. Men were forced to study the properties of a spherical surface in order to sail a very long distance away from the sight of land. With the first leaves there grew a symmetry that can be expressed only in geometric terms. When the bees needed a type of cell that would give them the greatest room for honey with the least amount of cell walls and no waste space, they chose a geometric form (hexagonal) for their

cells. They had been storing honey in hexagonal cells for untold centuries before the mathematicians proved that hexagons of the same size with sides and angles equal may be fitted together on a plane in such a manner as to completely cover it. As civilization developed men began to discover more and more of the geometric relations we know so well. You have learned formulas for the measure of the areas and volumes of plane and solid figures. Has it ever occurred to you that there was a time when men did not know these formulas?

From this introduction it is easy to lead up to the Rhind Papyrus and some of its erroneous rules for areas, which should be compared with our formulas for the same. The Egyptian value of $\pi$ $(16/9)^2$, the Hebrew value, 3, the Chinese value $\sqrt{10}$ and the Hindu value 142/45 should all be compared with the value we use.

The question arises at once, "Which method is the correct one?" or rather, "How do we know that our formulas are correct?" The study of geometry will convince us that we are right within the limit of errors of measurement. Its conclusions represent the result of centuries of study to improve and extend existing knowledge. Fact after fact was discovered and error after error corrected. The ancient Egyptians were satisfied when they had learned enough to resurvey their fields after each annual overflow of the Nile, which wiped out their boundaries. Because of this use of geometry the Greeks gave it its name which means literally "the measure of the earth". But the Greeks were not satisfied to accept the Egyptian mensuration formulas and other practical rules and go no farther. They tried to and did establish new facts and relations about triangles, squares, rectangles, circles, and other geometric figures. During the six centuries just preceding the Christian era they were very active. They studied the subject for its own sake rather than for the practical use of it. Some of the theory they developed has found applications in the centuries that followed. The things they did are the heritage of the centuries and on them is based a great part of our practical mathematics, modern architecture, and engineering, as well as much of our art.

Such an introduction will not only stimulate interest in the subject but will help the teacher to win the approbation of the class and enlist the hearty cooperation of the pupils in the coming adventure.

When the time comes in the course to begin proving theorems by the assumption of postulates and axioms the class should be told of Euclid and the *Elements*.

A story of the development of trigonometry can be given which is just as interesting to the class in trigonometry and a few

words as to the life and works of Descartes will be interesting news to the class in analytic geometry.

*A knowledge of the history of mathematics gives both pupil and teacher an appreciation of the value of the subject and its inseparable and vital connection with the development of civilization.*

Our number system and the operations on numbers, our fractions, our algebraic symbolism, our geometry, our calculus, all were developed as the need for them arose in the progress of civilization. It is only fair to the pupil to let him see this, whether he be a high school or college student. As teachers we should certainly be fully informed on this subject. If the student is taught the history of mathematical processes he will see that they were not handed down from a mountain of inspiration but are the fruits of laborious thinking in the long and tedious journey up from barbarism to civilization and were developed for the most part as necessity demanded. He will see the worth while contribution that geometry has made to scientific progress from the time the ancient "rope stretchers" of King Sesostris laid out the right angles of their land surveys along the banks of the turbulent Nile to the present day when Einstein, forming a theory to explain the mysteries of the universe, assumes that figures may *not* be moved about in space without change of shape or size. He will see its contributing influence on art and culture to such an extent that he should feel that it is as much worth while to know about it as to know about many other subjects in the curriculum that do not contribute to his ability to earn a greater wage immediately on leaving school. He will see what a heritage is his.

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The Bureau of Publications of Teachers College, 525 West 120th Street, New York City, still have copies of the Second and Third Yearbooks of the National Council on hand. These will soon be exhausted and many teachers to say nothing of libraries will be unable to secure them. Mr. C. M. Austin has taken it upon himself to dispose of 50 copies of the Fourth Yearbook as well as the others. Let us have a number of our members of the Council follow Mr. Austin's example.

# MATHEMATICS IN INDUSTRY AND RESEARCH [1]

By E. J. MARTIN

*Physicist General Motors Corporation Research Laboratories, Detroit, Michigan*

In discussing mathematics in industry and research I have in mind the rôle of the mathematical sciences in industrial work undertaken to improve the products of industry, assure satisfactory supplies of raw materials, lower manufacturing costs, predict as far as possible future demands in various divisions of the market, facilitate distribution, and prepare the way in advance for the creation of new products when it seems advisable. It is the purpose of this paper to illustrate the importance of mathematical training for the personnel selected to solve these problems.

To improve a device of any kind it is necessary to understand the principles upon which its operation depends. It is true that accidental discoveries sometimes play important parts in the development of industrial products, but even in such cases careful systematic analysis generally increases the value of such discoveries. Men who are fortunate enough to hit upon things accidentally must have sufficient training to recognize and analyze the underlying principles and apply them properly in making improvements or be left out of the picture by those that are properly trained. To increase their knowledge of the scientific principles underlying their products, the larger and more progressive concerns employ and thoroughly equip large staffs for engineering experiment and research work. Such groups are not merely clusters of individuals working in seclusion to make startling discoveries; on the contrary, they are carefully selected so that all branches of science bearing on industrial problems are well represented. They are usually backed by very good technical libraries and all the equipment they need. The research programs are carefully made out and the problems freely discussed in conferences. The reports on laboratory progress are sent to the right men in charge of production.

When one considers the complexity of some of the common

[1] Read before the Michigan Educational Association October 26, 1928.

commodities to which such thorough research methods are applied, it is easy to see that industry requires men well trained in the mathematical sciences and consequently in mathematics. For example, let us consider such articles as electrical generators and motors, telephones and loud speakers. Research on the problems connected with these items requires a thorough knowledge of electricity and magnetism, mechanics and acoustics. In the study of electrical refrigeration there is need for a knowledge of the above subjects plus thermo-dynamics. In this particular work extensive tests must be made to determine the thermal properties of materials for the purpose of keeping those purchased up to standard as well as to develop new ones.

In the automotive industry the problems are very numerous and worthy of the efforts of men equipped with the best scientific training our educational systems can give them. The testing work done by this industry to maintain the standards of the raw materials it buys, as well as the quality of its product, is worth mentioning by itself. An automobile with its occupants may be regarded as a small community on wheels. In devising tests that are representative of the unusual wear and tear to which the materials are subjected in one of these communities, there is opportunity for many trained men to show their originality as well as ingenuity. Such testing often requires instruments that cannot be bought; they must be developed by the staff and constructed in a shop equipped for such work. Materials that are used in manufacturing parts which do not involve the safety of the passenger are usually tested by sample; but as far as possible, materials that go into parts which are important from the safety standpoint are given a 100 per cent inspection. Besides being accurate and reliable, apparatus for such tests must be rugged and capable of quick operation. After the manufacturing process the parts themselves must be inspected; the more vital ones undergoing complete inspection, while the others are inspected by sample.

Wherever possible the inspection of important items does not stop at the surface, but is carried to the inner structure for the purpose of revealing hidden flaws, strains, lines of flow, grain structure, and so forth. To make a 100 per cent inspection of this type on finished parts it is necessary to develop test methods which do not in any way injure the articles. For such work industry makes use of electricity, magnetism, light and x-rays.

To find relationships between the phenomena in these branches of science and the properties of matter for which the articles are to be tesed, the engineer usually begins by searching the scientific literature thoroughly. He must know sufficient mathematics to read any of this literature intelligently and apply its teachings to his problem. Sometimes no information can be found bearing on the problem but this is not often the case. It does often happen, however, that the information obtained by reading is incomplete or of doubtful accuracy. In such a case the engineer must plan and carry out a research program which will yield him the necessary information upon which to base his tests.

Although after a test method has been thoroughly worked out and standardized, it becomes more or less routine, the man to whom it is turned over must know what it is all about. He should know when the apparatus is out of adjustment and be able to readjust it. He must tabulate his data truthfully and do this computation accurately. Even for the simplest tests the man must take the work seriously and know that the success of it depends entirely on him.

The data from routine testing are often the cause of improvements in manufacturing processes. If serious defects show up occasionally or if minor defects become chronic, the cause is usually hunted until found. If the fault lies with the manufacturing process the necessary change is made if possible. If the raw material is to blame the concern supplying it is notified and they usually eliminate the trouble. It sometimes happens that concerns supplying raw materials are not equipped to find reasons for defects in them. In such cases the concern buying the material often locates their trouble for them and gives what assistance is needed to eliminate it.

I have given considerable attention to the testing and standardization because they are absolutely necessary if industry is to hold the ground it gains by developmental research and also because its importance and magnitude are not usually realized by those not closely connected with it. Furthermore this field of activity gives employment to a large number of men having scientific training from technical high schools, cooperative schools, colleges and universities.

In developmental work industrial activities extend from the fields of applied science far into the realms of pure science. Al-

though I mention applied science and pure science in the same sentence, I do not mean that one is in any way superior to the other. In fact spanning the gap between the generalizations of pure science and their application to making this world a better place in which to live usually requires very scientific bridge-work. In this connection I take the liberty of quoting Dr. H. F. Moore of the University of Illinois who writes in *Science* as follows: "Let us not forget that while pure science furnishes us with generalizations of observed phenomena into concise, summarized statements which we call laws, and also gives us the root ideas for many of the appliances and processes of applied science, yet it is equally true that the study of practical problems has been the source of not a few of the root ideas of pure science." Industrial leaders believe that knowledge is power, especially if that knowledge is scientific. For this reason they appropriate large funds to encourage research work.

In research devoted to the improvement of even the common devices sold to the public, a great burden is again placed upon instrumentation. It is usually necessary to make measurements while the devices are actually operating under normal or experimental conditions. For instance in the automobile it is quite easy to observe how the timing, transmission and differential gears function when turned very slowly, but it requires highly delevoped apparatus to enable the engineer to actually see how they behave at high speeds. A cam follower follows the cam faithfully at low speeds so that the valve-lift curve is what the engineer intended it to be. At high speeds, however, the valve spring, due to lack of stiffness or to conditions of high speed resonance, may not be able to keep the cam follower pressed firmly against the cam at all times. Therefore a study of the entire valve gear at high speeds is necessary if the engineer is to know how the valve is performing. In studying the combustion characteristics of an engine the pressures developed during the cycle must be faithfully recorded at different engine speeds. Similar measurements must be made on the ignition system, the manifolds, the carbureter, the lubricating system, and so forth throughout the car. It is also necessary to study the performance of the machinery used in the manufacturing process itself.

The attempts of some concerns to satisfy the demands of the public for smoothness and quietness of operation in the devices which it purchases, have increased considerably the need of men

trained in the mathematical sciences. The results of sound and vibration measurements are so sensitive to the conditions under which the experiments are conducted that even if ready made apparatus could be purchased for such purposes, a knowledge of acoustics, sound, and vibrating bodies is necessary for success. Since these measurements usually require the use of specially constructed high-quality vacuum tube amplifiers, a knowledge of electrical circuits is also necessary. When one remembers that the laws governing sounding bodies and electrical circuits are expressed by equations which are often similar in form, it is easy to see the value of a good mathematical background for such work.

Generally speaking, there is, in research work a great demand for instruments capable of recording rapidly fluctuating phenomena. Such instruments usually constitute mechanical systems characterized by one degree of freedom, intertia, elasticity, and friction. Therefore a thorough understanding of the mathematical theory of such a system when subjected to an external force is valuable to a man intending to enter industrial research. This is equivalent to saying that all the courses leading up to and including the study of differential equations are equally valuable.

The mathematical training necessary for the large corps of draftsmen employed by industrial concerns is too well known to require discussion here. The same is also true with regard to chemists, metallurgists, and others whose activities in industrial work have long been known to the public.

In its attempts to predict the demands for its products in various divisions of the market, industry requires the services of men trained for statistical work. It is highly important that manufacturing facilities be adjusted in time to supply future market requirements. In doing this work it is usually necessary to accumulate great quantities of information, determine its reliability, digest it, and then make the proper deductions.

I now want to take the liberty of making a criticism of mathematics as it is commonly taught. You have doubtless heard this criticism many times already. In the teaching of mathematics the various theorems, functions, equations and so forth that go to make up the subject are seldom or never emphasized from the standpoint of their importance in solving the problems that are met with in applied science. I believe that preparing a man to improve the adjustment between himself and his physical en-

vironment is an activity worthy of any science. It is true that mathematics pure, and unattached to anything physical, is of great value both as a discipline and as a training for those desiring to become mathematicians. However, those who must use mathematics as a means of solving problems in applied science far outnumber those preparing to become mathematicians and I believe they deserve courses of instruction planned to meet their needs.

If the rôle played by mathematics in the achievements of the present day could be outlined in a simple and definite manner, it might increase interest in that science during the high school period. "Current Events" and "Moderater Topics" are used by the teaching profession to give life to Civics and History. A few columns, or a page, each week in either of these periodicals or in "Science News-Letter" or some similar publication, might well be devoted to giving life to mathematics. By outlining in a simple manner the problems that have been solved in developing the conveniences of our everyday life, the editor of such a page would have ample, up-to-date material to work with.

As far as I know, mathematics is one of the few sciences taught without the aid of laboratory work. While at present I do not feel that a laboratory course in mathematics is either feasible or desirable in the college or university, I do believe that in the high school such a course is both feasible and desirable. One might begin in a small way by re-building the laboratory course in physics. At present it is used only as a means of emphasizing the principles of physics; all mathematical operations being made as few and as simple as possible. The laboratory course in physics might well be planned to bring out the importance and utility of the principles of high school mathematics. The new course would have to be taught either by a physics teacher with teaching ability in mathematics or by a mathematics teacher with teaching ability in physics. I believe such teachers can easily be found or developed.

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# NEWS NOTES

The thirteenth annual meeting of the Mathematical Association of America was held at Columbia University, New York, New York, on Friday and Saturday, December 28–29, 1928, in affiliation with the American Association for the Advancement of Science and the American Mathematical Society. The Society held sectional sessions on Thursday at Milbank Hall, Barnard College, a general session on Friday morning at which the third Bôcher Memorial prize was awarded, a joint session on Friday afternoon with Section A and the Mathematical Association, a joint session on Saturday morning with Section K of the A. A. A. S.

On Friday evening there was a joint dinner for attending mathematicians and their guests in the Men's Faculty Club, Columbia University.

The program of the Mathematical Association was as follows:

*Friday Afternoon*

1. "The relation of Statistics to Modern Mathematical Research," Professor DUNHAM JACKSON, University of Minnesota, retiring vice-president of Section A of the American Association.
2. "The Heroic Age of Geometry," Professor J. L. COOLIDGE, Harvard University, representing the Mathematical Society of Mathematical Association.

*Saturday Morning*

1. "A Simple Principle of Unification in the Elementary Theory of Numbers," Professor R. D. CARMICHAEL, University of Illinois.
2. "Geometry of Functions of Two Complex Variables," Professor EDWARD KASNER, Columbia University.
3. Report on the Bologna Congress, Professor H. W. TYLER, Massachusetts Institute of Technology. Remarks by delegates.
4. Annual Business Meeting and Election of Officers.

*Saturday Afternoon*

1. "The Life Insurance Actuary and His Mathematics," RAYMOND V. CARPENTER, Actuary, Metropolitan Life Insurance Company, by invitation.
2. "The Development of Mathematics in Sweden," Professor EINAR HILLE, Princeton University.
3. "Selected Topics in Calculus for High Schools," Mr. JOHN SWENSON, Wadleigh High School, New York City, by invitation.

Believing that our readers will be interested in the program of the Annual Meeting of The Mathematical Association of Great Britain we are reproducing below the announcement for their recent meeting.

THE MATHEMATICAL ASSOCIATION

President: W. F. SHEPPARD, Sc.D., LL.M.

The ANNUAL MEETING of the Mathematical Association will be held at the London Day Training College, Southampton Row, London, W.C. 1, on MONDAY and TUESDAY, 7th and 8th JANUARY, 1929.

*Monday Afternoon, 7th January, 1929*

1. 3.15 P.M.—BUSINESS.
   (*a*) The Report of the Council for the year 1928.
   (*b*) The Treasurer's Report for the year 1928.
   (*c*) The Election of a President for the year 1929.

   The Council nominate DR. W. F. SHEPPARD for re-election.

   (*d*) The Election of Officers and Council for the year 1929.

   The Council nominate MR. W. HOPE-JONES, B.A., and MISS E. WISE in place of MISS M. A. HOOKE and MR. N. M. GIBBINS who retire in compliance with the Rules.

   (*e*) Other business (if any).
2. 4.00 P.M.—"The Axioms of Geometry," by H. G. FORDER, B.A. (Hymers College, Hull).
3. 4.45 P.M.—Interval for tea, which will be provided at 9d. per head.
4. 5.30 P.M.—"Modern Mathematical Problems in Aerodynamics," by PROFESSOR H. M. LEVY, D.Sc., F.R.S.E. (Imperial College of Science and Technology). Lantern.

   The lines of approach taken by mathematicians in the study of Aerodynamical problems; and the fields in which future research is most urgently required.

*Tuesday Morning, 8th January, 1929*

5. 10.00 A.M.—Discussion: "Should a candidate for School Certificate be allowed to take, in place of the Mathematics and Science Group, a group

containing Drawing and Music and possibly other subjects?''

To be opened by MISS E. R. GWATKIN, M.A. (Head Mistress of Streatham Hill High School, S.W. 2).

The regulations for the various School Certificate examinations differ in detail, but are alike in general principle. The subjects fall into four groups: I—English and English subjects; II—Languages; III—Mathematics and Science; IV—Drawing, Music and, in some cases, other subjects. To obtain a certificate, a candidate must satisfy certain conditions, one of which is that he must pass in each of the first three groups. The Association of Head Mistresses have recently urged that a candidate should be allowed to take Group IV in place of either Group II or Group III. This, so far as it relates to Group III, is the subject for discussion.

NOTE.—*Members who would like to take part in the discussion are invited to send their names to to one of the Honorary Secretaries, stating on which side they would speak.*

6. 11.30 A.M.—Interval.

7. 11.45 A.M.—''The use and abuse of formulæ,'' by N. J. CHIGNELL, M.A. (Charterhouse).

*Summary.*—(A) Two purposes in the use of formulæ: (*a*) calculation, (*b*) construction. Manipulation or transformation intermediate between these. All these purposes arrived at by intelligent translation of the language of the formula.

(B) (1) Distinction between (*a*) and (*b*) to be strictly observed. (2) Formulæ not to be used for any purpose other than mere calculation unless the process leading to the formula is so well established that a pupil can construct the formula himself. Exceptions.

(C) Transformation of formulæ treated as a reversal of processes.

*Tuesday Afternoon, 8th January, 1929*

8. 2.30 P.M.—THE PRESIDENT: ''Variety of Method in the Teaching of Arithmetic.''

9. 3.30 P.M.—Interval.

10. 3.45 P.M.—''Methods of Voting in Theory and in Practice,'' by Professor J. E. A. STEGGALL, M.A., F.R.S.E. (The University of St. Andrews).

NOTE.—*A Publishers' Exhibition will be arranged in connection with the meeting and will be open on both days.*

*Honorary Secretaries:*

C. PENDLEBURY, M.A.,
39 Burlington Road, Chiswick, London, W. 4.

MARGARET PUNNETT, B.A.,
The London Day Training College,
Southampton Row, W.C. 1.

# NEW BOOKS

**Exercises and Tests in Plane Geometry**. By DAVID EUGENE SMITH, WILLIAM DAVID REEVE, and EDWARD LONGWORTH MORSS. Ginn and Company, 1928. Price: $0.48.

The *Exercises and Tests in Plane Geometry* resembles the companion volume for algebra in its general form: the abundance of tests, the perforated pages, the record blank, and the explanations given in the *Teacher's Manual*. The difference are those necessitated by the differences in subject matter. The technique of geometry are not the same as those of algebra and they cannot be treated in the same way.

In some cases, the geometry tests are given in pairs. More frequently, the same idea will appear later in the book, using different geometric subject matter. There is greater variety in the form of the questions than in the algebra booklet. In fact, every sort of new type question seems to be included.

The booklet is proving exceedingly useful for its introduction to geometry and it effectively solves the question of quiz material for the early part of the work. Furthermore, whether assigned as home work, class work, or examinations, the student is brought to the writing of a formal proof in a way that guards against the forming of bad habits of presentation and that facilitates pendant work. The student is asked to select relevant data, to supply reasons for statements, to determine necessary and sufficient conditions for a proof, and the like.

It is unfortunate that the form of the questions makes the scoring difficult in some of the tests, but it is difficult to see how the desired ends could have been obtained by other types of questions. The construction problems are especially good. One page, for example, asks for the construction of a line through a given point and perpendicular to a given line in each of eleven different cases. The lines are vertical, horizontal and oblique. At the close of this exercise, few pupils will suppose that all perpendiculars must be vertical lines, or at least parallel to the edges of the blackboard.

The work calls for a great amount of discriminating thinking, but perhaps the most significant test of the booklet is that students like it—they appreciate its definiteness and its clarity.

VERA SANFORD

**Functions of Real Variables.** E. J. TOWNSEND, Ph.D., University of Illinois (Henry Holt & Co., Publishers, 1928, 405 pages).

The admirers of Professor Townsend's "Theory of Functions of a Complex Variable" have been waiting for this companion text in the theory of real variables. In the author's characteristically clear style of exposition, this new treatise covers the following topics:

Real Number System,
Theory of Point Sets,
Continuity and Discontinuity of Functions,
Derivatives and Their Properties,
Riemann Theory of Integration,
Lebesque and other Integrals, and
Infinite Series.

H. A. DOBELL

N. Y. STATE COLLEGE FOR TEACHERS

**A First Course in the Differential and Integral Calculus.** W. B. FORD, University of Michigan (Henry Holt & Co., Publishers, 1928, 372 pages).

For the most part this text covers the usual material as presented in a first course. Its seventeen chapters are headed:

Fundamental Notions and Definitions,
The Derivative of a Function,
General Theorems Concerning the Derivative,
Maxima and Minima, Points of Inflection,
Further Applications of the Derivative to Geometry,
Rates,
Partial Derivatives,
Singular Points, Asymptotes, Curve Tracing,
Law of the Mean, Indeterminate Forms,
Infinite Series,
Infinitesimals, Differentials,
Integration,
Applications of Integration,
The Definite Integral,
The Definite Integral as the Limit of a Sum,
Successive Integration, and
Differential Equations.

In an Appendix the method of least squares is developed and applied to curve fitting. This is a timely addition in the light of present day interest in statistics. The book is concluded by integral and trigonometric tables, formulas from trigonometry, tables of squares, cubes, etc., and diagrams of important curves.

The reviewer is impressed by the care which has been taken in the selection of problems. On this point the author says, "By actual count there are 2,005 examples and exercises. Of these, 179 are worked in full and serve as illustrative material complete in all particulars. Of the remainder, 89 are accompanied by both the answer and a hint as to the method of solution, 1,132 are accompanied by the answer only, while, finally, 605 have neither answer nor hint given. This unusually large total of varied illustrations and exercises, coupled with the care with which they have been arranged and graded, should enable the text to meet all requirements in this regard." Further adaptability is sought by the addition of numerous physical applications.

H. A. DOBELL

N. Y. STATE COLLEGE FOR TEACHERS

**Calculus.** By H. B. FINE. The Macmillan Co., 1927. 418 pages.

With the untimely death of Professor H. B. Fine of Princeton Uni-

versity, in December, 1928, America has lost a great teacher. The fact that he was a great teacher can in no way be better exemplified than by his excellent text in Calculus that has been recently published by the Macmillan Company. One feels on reading the book that there is nothing a pupil will have to unlearn in later courses in mathematics, and that he will carry away a greater respect for rigor than can oftentimes be obtained from other texts.

The book has, in addition to the usual material found in an elementary text, chapters on Fourier series and function of a complex variable. Even with this additional material the text is not too condensed, and the exposition is clear. This book should commend itself to mathematics teachers all over the country.

G. R. MIRICK

**Ziffern und Ziffernsysteme.** I. TEIL. By EUGEN LÖFFLER. Leipzig, Teubner, 1928. Pp. ii + 54.

This little work by Dr. Löffler is already well known by those who are familiar with the literature of the history of mathematics, having heretofore appeared in two editions. It speaks well for its reception that the first part of the second edition is now republished, with such corrections as have been found essential. The periods covered are those of (1) the origins of number writing, (2) the Egyptian symbols, (3) the numerals of the Babylonians and Assyrians, (4) those of the Greeks, and (5) those of the Semitic peoples. The first of these treats of the philosophic aspects of number writing, and of the probable origins of the scales of notation and the decimal system. The author then discusses the forms of the numerals as found among the early civilizations. Of the Egyptian numerals he gives in facsimile the hieroglyphic symbols, the hieratic (those used in the Ahmes papyrus), and the demotic. The Babylonian and Assyrian forms are limited to the simpler ones, these being quite sufficient for the purposes. The three common forms of the Greek numerals are given, together with a table showing the relation of the second Greek system to the Semitic letters—an interesting relationship not ordinarily given in our histories. This chapter also includes the fractional notation of the Greeks, with some mention of the approach to a place value.

The chapter on Semitic numerals takes up a less familiar field of study. It is illustrated by facsimiles of the Phoenician, Palmyrean, Syriac, and Hebrew forms, and the author calls attention of the dependence of the Ethiopian, Arabic, Coptic, Armenian, Persian, and Turkish numerals upon these earlier forms. It seems from his studies that the Hebrew alphabetic numerals go back no farther than 137 B.C., the use of the finals for *kaph*, *mem*, *nun*, *pe*, and *zade* for 500–900 being considerably later.

The bibliography is limited almost entirely to German sources, with mention of four English works, one French, and one Latin. The work pretends to be no more than a brief summary of the history of early number forms, and as such it will be helpful to beginners in the study of the subject.

DAVID EUGENE SMITH

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